Estimation Bounds for Localization

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Abstract— **The localization problem is fundamentally important for sensor networks. We study the Cramér-Rao lower bound (CRB) for two kinds of localization based on noisy range measurements. The first is Anchored Localization in which we know true positions of at least 3 nodes. We show some basic invariances of the CRB in this case and derive lower and upper bounds on the CRB which can be computed using only local information. The second is Anchor-free Localization where no absolute positions are known. Although the Fisher Information Matrix is singular, we derive a CRB-like bound on the total estimation variance. Finally, for both cases we discuss how the bounds scale to large networks under different models of wireless signal propagation.**

I. Introduction

In wireless sensor networks, the positions of the sensors play a vital role. Position information can be exploited within the network stack at all levels from improved physical layer communication[1] to routing[2] and on to the application level where positions are needed to meaningfully interpret any physical measurements the sensors may take. Because it is so important, this problem of localization has been studied extensively. Most of these studies assume the existence of a group of "anchor nodes" that have *a-priori* known positions. There are three major category of localization schemes that differ in what kind of geometric information they use to estimate the locations. Many[3], [4], [5], [6] and [7], use only the connectivity information based on whether node i can directly communicate with node j, or anchor k. Such approaches are attractive because connectivity information is easily accessible at the network layer where it is also essential for multi-hop routing.

The second category uses both Euclidean distances and angular information for localization. Such schemes are studied in [8], [9] and [10]. These are useful when antenna arrays are available at the sensor nodes so that beamforming is possible in order to determine what direction signals are coming from.

The third category is localization based solely on the Euclidean distances (ranging) between nodes and between nodes and anchors [11], [12], [13] and [14]. Within these, the first two papers cited estimate the coordinates of the nodes based on the estimated distances between the nodes and the anchors directly. The latter two schemes first estimate positions in an anchor-free coordinate system and then embed it into the coordinate system defined by the anchors. In [15] and [16], the schemes for estimating Euclidean distances are discussed. In this paper we focus on this kind of localization problem using ranging information.

The Cramér-Rao lower bound (CRB) [17] is widely used to evaluate the fundamental hardness of an estimation problem. The CRB for anchored localization using ranging information has been studied in [18] and [19] for several specific geometric setups. For anchor-free localization, as mentioned in [9], the FIM is singular and so the standard CRB analysis fails.[20]

A. Outline of the paper

After reviewing some basics, in Section II we study estimation bounds for anchored localization. Assuming the ranging errors are iid Gaussian, we give an explicit expression for the FIM solely based on the geometry of the sensor network and show that the CRB is invariant under zooming and translation. Rotation does not change the lower bound on $E((x_i - \hat{x}_i)^2) + (y_i - \hat{y}_i)^2)$. Using matrix theory, we give a lower bound on the CRB which is determined by only local geometry. This converges to the CRB if the local area is expanded. We also give an upper bound on localization performance using only local information. Finally we study the wireless situation in which the noise variance on the range measurements depends on the inter-sensor distance. Simulation results validate our intuition that the faster the signal decays, the less the estimation bound benefits from faraway information. A heuristic argument is given that reveals the basic scaling laws involved.

In Section III we study the estimation bounds for anchor-free localization. We show that the rank of the FIM is at most $D_{FIM}-3$, where D_{FIM} is the dimension of the matrix. We give a bound on the total estimation variance in the anchor-free estimation and observe that the per node bound in simulations appears proportional to the average number of neighbors. We conjecture that the average estimation variance depends on the received signal energy per node.

B. Cramér-Rao bound on ranging

Since range is our basic input, we first review the CRB for wireless ranging. The distance between two nodes is $t_d c$, where c is the speed of light and t_d is the time of arrival (TOA). TOA estimation is extensively studied in the radar literature. If T is the observation duration, $A(t)$ is the pulse[1], and N_0 is the noise power spectral density, then for any unbiased estimate of t_d [21]:

$$E[(\hat{t}_d - t_d)^2] \geq \frac{N_0}{\int_0^T [\frac{\partial A(t)}{\partial t}]^2 dt}$$

Notice that $\int_0^T (\frac{\partial A(t)}{\partial t})^2 dt$ is proportional to the energy in the signal where the proportionality constant depends on the shape of the pulse. Because of the derivative, we know that having significant pulse energy at high frequencies (i.e. a signal with wide bandwidth) is beneficial for localization. Calling that proportionality τ_r^2 we have:

$$E[(\hat{t}_d - t_d)^2] \geq \frac{\tau_r^2}{SNR} \tag{1}$$

C. Models of Localization

In idealizing the localization problem, we assume all the sensors are fixed on a 2-D plane. We have a set S of M sensors with unknown positions, together with a set F of N sensors (anchors) with known positions. Because the size of each sensor is assumed to be very small, we treat each sensor as a point.

Each sensor generates limited-energy wireless signals, by means of which node i can measure the distance to some nearby sensors in the set $adj(i)$. We assume $j \in adj(i)$ iff $i \in adj(j)$.[2] Throughout, we also assume high SNR[3] and so are free to assume that the distance measurements are only corrupted by independent zero mean Gaussian noises.

1) Anchored Localization Problem: If there are at least three nodes with known positions ($|F| \geq 3$), then it is possible to estimate absolute coordinates for each node using observations D and position knowledge P_F.

$$D = \{\hat{d}_{i,j} | i \in S \cup F, j \in adj(i)\} \tag{2}$$
$$P_F = \{(x_i, y_i)^T | i \in F\} \tag{3}$$

Our goal is to estimate the set

$$P_S = \{(\hat{x_i}, \hat{y_i})^T | i \in S\} \tag{4}$$

(x_i, y_i) is the position of a single sensor i. $\hat{d}_{i,j}$ is the measured distance between sensor i and j. $\hat{d}_{i,j} = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2} + \epsilon_{i,j}$, where $\epsilon_{i,j}$'s are modeled as independent additive Gaussian noises $\sim N(0, \sigma_{ij}^2)$.

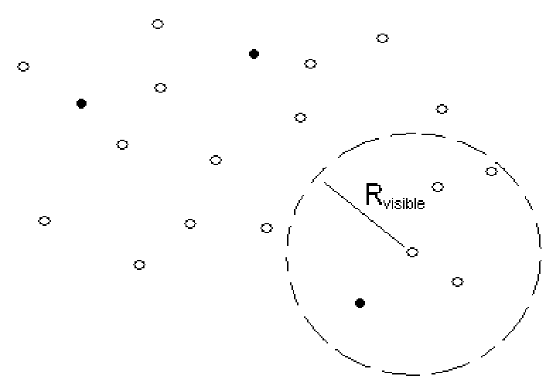

Fig. 1. A sensor network, solid dots are anchors, circles are nodes with unknown positions. The ranging information $\hat{d}_{i,j}$ is only known for sensor pair i, j s.t. $d_{i,j} \leq R_{visible}$.

[1]Notice that we can get ranging estimates from any pulse whose shape is known at the receiver. This can include a data carrying packet that has been successfully decoded as long as we know the time it was supposed to have been transmitted. In a wireless sensor network, we are not necessarily restricted to use a radio that is dedicated only to the function of ranging.

[2]In general, if node i can measure the distance to node j that does not necessarily mean that node j can measure the distance to node i as well. However, we will assume that sensor i and j can communicate with each other and hence the distance between i and j is known to both. Also, if i and j get different distance measures, we assume that they have been appropriately averaged together.

[3]Suppose that we are estimating the propagation time by looking for a peak in a matched filter. By high SNR we mean that the peak we find is in the near neighborhood of the true peak. At low SNR, it is possible to become confused due to false peaks arising entirely from the noise.

2) Anchor-free Localization Problem: If $|F| = 0$, no nodes have known positions. This is an appropriate model whenever either we do not care about absolute positions, or if whatever global positions we do have are far more imprecise than the quality of measurements available within the sensor network. If $P_S = \{(\hat{x_i}, \hat{y_i})^T | i \in S\}$ is an estimate of node positions, then $P'_S = \{R(\alpha)(\pm\hat{x_i}, \hat{y_i})^T + (a,b)^T | i \in S\}$ is equivalent to P_S where the $\pm$ represents reflecting the entire network about the y axis and $R(\alpha)$ is a rotation matrix:

$$R(\alpha) = \begin{pmatrix} cos(\alpha) & -sin(\alpha) \\ sin(\alpha) & cos(\alpha) \end{pmatrix} \tag{5}$$

As a result, the performance measure of an anchor-free localization should not be $\sum_i (x_i - \hat{x_i})^2 + (y_i - \hat{y_i})^2$. Instead we will use the distance between equivalence classes. Since the FIM for anchor-free localization is singular [9], we will develop a bound using the tools provided in [20].

II. Estimation Bounds for Anchored Localization

For the Anchored Localization problem, the Cramér-Rao bound (CRB) can be directly derived from the Fisher Information Matrix (FIM).

A. The FIM for Anchored Localization

As illustrated in Fig.2, we define $\alpha_{ij} \in [0, 2\pi)$, the angle from node i to j, as:

$$\cos(\alpha_{ij}) = \frac{x_j - x_i}{\sqrt{(x_j - x_i)^2 + (y_j - y_i)^2}}; \qquad \sin(\alpha_{ij}) = \frac{y_j - y_i}{\sqrt{(x_j - x_i)^2 + (y_j - y_i)^2}}; \tag{6}$$

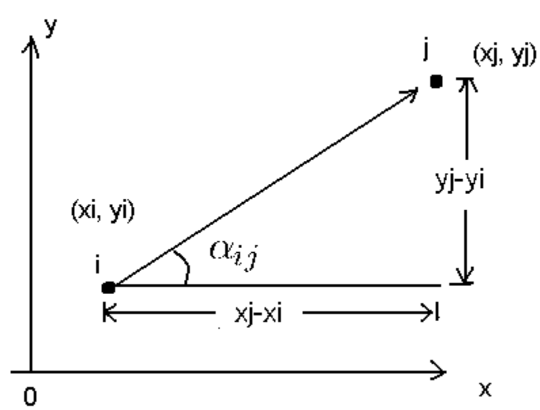

Fig. 2. α_{ij}

Let x_i, y_i be the $2i-1$'th and $2i$'th parameters to be estimated respectively, $i = 1, 2, \ldots, M$. The FIM is $J_{2M \times 2M}$.

Theorem 1: FIM for Anchored Localization $\forall i = 1, \ldots, M$

$$J_{2i-1,2i-1} = \sum_{j \in adj(i)} \frac{\cos^2(\alpha_{ij})}{\sigma_{ij}^2} \tag{7}$$

$$J_{2i,2i} = \sum_{j \in adj(i)} \frac{\sin^2(\alpha_{ij})}{\sigma_{ij}^2} \tag{8}$$

$$J_{2i-1,2i} = J_{2i,2i-1} = \sum_{j \in adj(i)} \frac{\cos(\alpha_{ij})\sin(\alpha_{ij})}{\sigma_{ij}^2} \tag{9}$$

For nondiagonal entries $j \neq i$, if $j \in adj(i)$:

$$J_{2i-1,2j-1} = J_{2j-1,2i-1} = -\frac{1}{\sigma_{ij}^2}\cos^2(\alpha_{ij}) \tag{10}$$

$$J_{2i,2j} = J_{2j,2i} = -\frac{1}{\sigma_{ij}^2}\sin^2(\alpha_{ij}) \tag{11}$$

$$J_{2i-1,2j} = J_{2j,2i-1} = J_{2i,2j-1} = J_{2j-1,2i} = -\frac{1}{\sigma_{ij}^2}\sin(\alpha_{ij})\cos(\alpha_{ij}) \tag{12}$$

If $j \notin adj(i)$, the entries are all zero.

Proof: : We have the conditional pdf:

$$p(\vec{d}|x_1^M, y_1^M) = \prod_{i<j, j \in adj(i)} \frac{e^{\frac{-(\hat{d}_{ij} - d_{ij})^2}{2\sigma_{ij}^2}}}{\sqrt{2\pi\sigma_{ij}^2}}$$

The Log-likelihood is $\ln(p(\vec{d}|x_1^M, y_1^M)) = C - \sum_{i<j, j \in adj(i)} \frac{(\hat{d}_{i,j} - d_{i,j})^2}{2\sigma_{ij}^2}$ and so:

$$\begin{aligned} J_{2i-1,2i-1} &= E\left(\frac{\partial^2 \ln(p(\vec{d}|x_1^M, y_1^M))}{\partial x_i^2}\right) \\ &= \sum_{j \in adj(i)} \frac{1}{\sigma_{ij}^2}\left(\frac{x_j - x_i}{\sqrt{(x_j - x_i)^2 + (y_j - y_i)^2}}\right)^2 \\ &= \sum_{j \in adj(i)} \frac{\cos^2(\alpha_{ij})}{\sigma_{ij}^2} \end{aligned}$$

and similarly for other entries of J. □

B. Properties of the CRB of Anchored Localization

Given the FIM, we can evaluate the CRB for any unbiased estimators:

$$\begin{aligned} E((\hat{x_i} - x_i)^2) &\geq J^{-1}_{2i-1,2i-1} \\ E((\hat{y_i} - y_i)^2) &\geq J^{-1}_{2i,2i} \end{aligned}$$

Corollary 1: The FIM is invariant under zooming and translation $J(\{(x_i, y_i)\}) = J(\{(ax_i + c, ay_i + d)\})$ for $a \neq 0$.

Proof: : The angles α_{ij} and noise σ_{ij} are unchanged and so the result follows immediately. □

Corollary 2: The CRB for a single node is invariant under rotation and reflection: Let $A = J(\{(x_i, y_i)\})$, $B = J(\{R(x_i, y_i)\})$, where R is a 2×2 matrix, with $RR^T = I_{2\times 2}$. Then $A^{-1}_{2i-1,2i-1} + A^{-1}_{2i,2i} = B^{-1}_{2i-1,2i-1} + B^{-1}_{2i,2i}, \forall i = 1, 2 \ldots, M$.

Proof: : Going through the derivation of the FIM, we find that $B = QAQ^T$, where Q is a $2M \times 2M$ matrix with the following form:

$$\begin{pmatrix} Q_{2i-1,2i-1} & Q_{2i-1,2i} \\ Q_{2i,2i-1} & Q_{2i,2i} \end{pmatrix} = R \tag{13}$$

with all other entries of Q being 0. Obviously $Q^TQ = QQ^T = I_{2M\times 2M}$. $B = QAQ^T$ and so $B^{-1} = QA^{-1}Q^T$. Write

$$\begin{pmatrix} A^{-1}_{2i-1,2i-1} & A^{-1}_{2i-1,2i} \\ A^{-1}_{2i,2i-1} & A^{-1}_{2i,2i} \end{pmatrix} = A(i) \tag{14}$$

Similarly for $B(i)$, then $B(i) = RA(i)R^T$. Notice that for any two matrices X and Y, $Tr(XY) = Tr(YX)$, then we have: $B^{-1}_{2i-1,2i-1} + B^{-1}_{2i,2i} = Tr(B(i)) = Tr(RA(i)R^T) = Tr(R^TRA(i)) = Tr(A(i)) = A^{-1}_{2i-1,2i-1} + A^{-1}_{2i,2i}, \forall i = 1, 2, \ldots, M$. □

C. A lower bound on the CRB for anchored localization

In order to evaluate the CRB, we need to take the geometry of the whole sensor network into account. In this section, we derive a performance bound for node l that depends only on the local geometry around it. This has the potential to be valuable in any "local" algorithms that try to do localization without performing all the computations in one center.

First we review a lemma for estimation variance:

Lemma 1: Submatrix bound Let $\theta = (\theta_1, \theta_2, \ldots, \theta_N) \in R^N$, $\forall M, 1 \leq M < N$, write $\theta^* = (\theta_{N-M+1}, \ldots, \theta_N)$, then for any unbiased estimator for θ,

$$E((\theta^* - \hat{\theta^*})^T(\theta^* - \hat{\theta^*})) \geq C^{-1} \tag{15}$$

Where C is the $(N - M) \times (N - M)$ matrix :

$$J(\theta) = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix} \tag{16}$$

where $J(\theta)$ is the non-singular, and hence positive definite, FIM for θ.

Proof: : Write the inverse of $J(\theta)$ as :

$$J(\theta)^{-1} = \begin{pmatrix} A' & B' \\ B'^T & C' \end{pmatrix} \tag{17}$$

$J(\theta)$ is positive definite, then

$$C' \geq C^{-1} \tag{18}$$

The proof of this is in [22] and the CRB theorem then gives $E((\theta^* - \hat{\theta^*})^T(\theta^* - \hat{\theta^*})) \geq C' \geq C^{-1}$. □

Notice that for any subset of M parameters, we can always change the index of the parameter to make them have index $N - M + 1, \ldots, N$. By directly applying Lemma 1 we get:

Theorem 2: A lower bound on the CRB
Write $\theta_l = (x_l, y_l)^T$ and write

$$J_l = \frac{1}{\sigma^2}\begin{pmatrix} J(\theta)_{2l-1,2l-1} & J(\theta)_{2l-1,2l} \\ J(\theta)_{2l,2l-1} & J(\theta)_{2l,2l} \end{pmatrix} \tag{19}$$

Then for any unbiased estimator $\hat{\theta}$. $E((\hat{\theta}_l - \theta_l)(\hat{\theta}_l - \theta_l)^T) \geq J_l^{-1}$.

Corollary 3: Let $\theta = \{x_1, y_1, \ldots, x_M, y_M\}$, J_l defined in the previous theorem is only dependent on (x_l, y_l) and $(x_i, y_i), i \in adj(l)$. In other words, we can give a performance bound on the estimation of (x_l, y_l) using only the local geometry around sensor l.

Proof: : Only need to notice that J_l in Eqn.7 only depends on $(\alpha_{lj}, \sigma_{lj})$, $j \in adj(l)$. These only depend on (x_l, y_l) and (x_i, y_i). □

We now assume that the ranging errors are i.i.d. Gaussian with zero mean and common variance σ^2 and define the normalized Fisher information matrix $K = \sigma^2 J$. It is similar to the Geometric Dilution of Precision (GDOP) in the radar literature[23] in that K is a dimensionless value only depending on the angles α_{ij}'s. Let $W = |adj(l)|$ with sensors $\in adj(l)$ being $l(1), \ldots, l(k), \ldots, l(W)$. Using elementary trigonometry and writing $\alpha_k = \alpha_{l,l(k)}$:

$$J_l = \frac{1}{\sigma^2}\begin{pmatrix} \frac{W}{2} + \frac{\sum_{k=1}^{W} cos(2\alpha_k)}{2} & \frac{\sum_{k=1}^{W} sin(2\alpha_k)}{2} \\ \frac{\sum_{k=1}^{W} sin(2\alpha_k)}{2} & \frac{W}{2} - \frac{\sum_{k=1}^{W} cos(2\alpha_k)}{2} \end{pmatrix} \tag{20}$$

The sum of the estimation variance

$$E((x_l - \hat{x_i})^2 + (y_l - \hat{y_i})^2) \geq {J_l^{-1}}_{11} + {J_l^{-1}}_{22} = \frac{4W\sigma^2}{W^2 - (\sum_{k=1}^{W} cos(2\alpha_k))^2 - (\sum_{k=1}^{W} \sin(2\alpha_k))^2} \geq \frac{4\sigma^2}{W} \tag{21}$$

taking equality when $\sum_{k=1}^{W} \sin(2\alpha_k) = 0$, $\sum_{k=1}^{W} \cos(2\alpha_k) = 0$. This happens if the centroid of the unit vectors $(\cos(2\alpha_k), \sin(2\alpha_k))$'s is at the origin $(0,0)$. A special case is when $\alpha_k = \frac{2k\pi}{W} + \beta$ and the angles $2\alpha_k$'s are uniformly distributed in $[0, 2\pi)$.

Above, we used **one-hop** geometric information around node i to get a lower bound on the CRB. This bound can be interpreted as the CRB given perfect knowledge of the positions of **all** other nodes. We can use more information to get a tighter bound on the CRB. The lower bound using 2-hop information is the CRB given the positions of **all** nodes j, $j \notin adj(i)$, and similarly for multiple-hops. The larger the local region we use to calculate the CRB, the tighter it is.

In our simulation, we have 200 nodes and 10 anchors all uniformly randomly distributed inside the unit circle, $j \in adj(i)$, if and only if $d_{i.j} \leq 0.3$. In Figure. 3, we plot the bounds for 20 randomly chosen nodes.

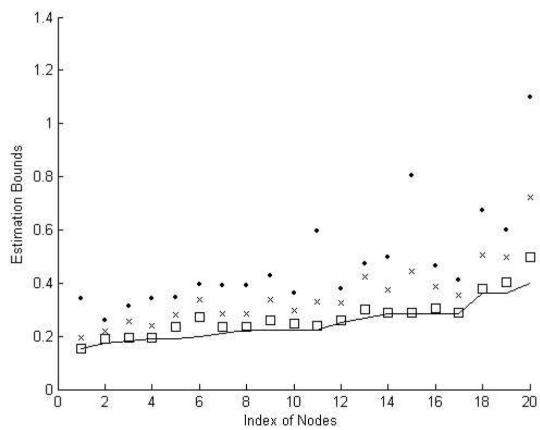

Fig. 3. Dots: CRB, Crosses: 2-hop bound, Squares: 1-hop bound, Curve: $\frac{4}{adj(l)}$
The nodes are indexed with decreasing $adj(i)$

D. An upper Bound on the CRB of anchored localization

The CRB bound in Theorem 1 gives us the best performance an unbiased estimator can achieve given **all** information from the sensor network, including the positions of all anchors and all the available ranging information $\hat{d}_{i,j}$. This bounds the performance of a centralized localization algorithm where a central computer first collects all the information and then estimate the positions of the nodes.

In a sensor network, distributed localization is often preferred due to the potentially lower communication and computation costs. In this "local" estimation problem only a subset of the anchors $F_l \subseteq F$ and a neighborhood of the nodes $l \in S_l \subseteq S$ may be taken into account. Given the positions of some of the anchors in F_l and the distances between some of the node pairs in S_l, $D_l = \{d_{i,j} | i \in S_l \cup F, j \in adj(i) \cap S_L\}$, our goal is to estimate the x_l, y_l. The CRB $V(x_l)$ and $V(y_l)$ of this estimation problem computed from the $2|S_l| \times 2|S_l|$ FIM is an upper bound on the CRB directly computed from the $2M \times 2M$ FIM because strictly less information is used for estimation. In this section, we compare the two bounds through simulation.

The wireless sensor network is shown in Fig.4. Anchors are on the integer lattice points in a 7×7 square region. There are 20 nodes with unknown positions uniformly randomly distributed inside each grid square. Sensor pair i and j can see each other only if they are separated by a distance less than 0.5. In the figure, we plot the nodes S_A inside the central grid in black, those inside $B_1B_2B_3B_4$ in gray, and those nodes inside $C_1C_2C_3C_4$ in light gray.

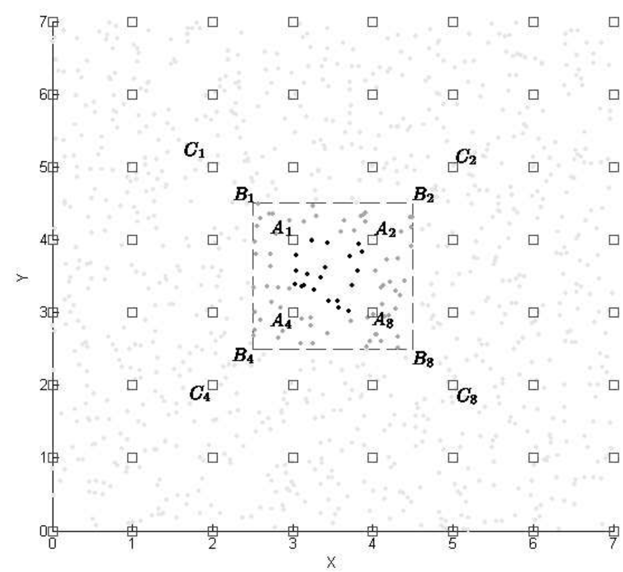

Fig. 4. The setup of the sensor network
Anchors are shown as squares, nodes are shown as dots, nodes inside the central grid are shown as black dots.

We compute the normalized CRBs ($V_i = V_i^x + V_i^y, i = 1, 2, \ldots, 20$) on the estimation of the positions of the nodes inside the central grid $A_1A_2A_3A_4$ in 4 different cases corresponding to information from within the squares: $A_1A_2A_3A_4$, $B_1B_2B_3B_4$, $C_1C_2C_3C_4$, and the whole sensor network. As shown in Fig.5, $V_i(A) \geq V_i(B) \geq V_i(C) \geq V_i(ALL), i = 1, 2, \ldots, 20$. We observe that $V_i(C)$ (squares in Fig.5) is extremely close to $V_i(ALL)$ (the curve in Fig.5). More surprisingly, we observe that $V_i(B)$ is much smaller than $V_i(A)$.

To explore further, we gradually increase the size of the square region which contains $A_1A_2A_3A_4$ in the middle, and compute the average CRB. As shown in

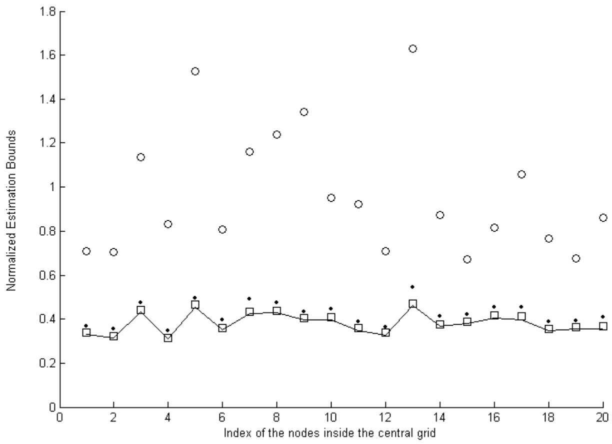

Fig. 5. Cramér-Rao bounds
Circle: estimation bounds using the information inside $A_1A_2A_3A_4$.
Dot: estimation bounds using the information inside $B_1B_2B_3B_4$.
Square: estimation bounds using the information inside $C_1C_2C_3C_4$.
Curve: estimation bounds using all the information.

Fig.6, the average CRB decreases as the size increases. Notice that $A_1A_2A_3A_4$, $B_1B_2B_3B_4$, $C_1C_2C_3C_4$ are the square regions with size 1, 2, 3 respectively. After first dropping significantly, the upper bound levels off significantly once we have included all the nodes directly adjacent to our neighborhood. This bodes well for doing localization in a distributed fashion — distant anchors and ranging information does not significantly improve the estimation accuracy.

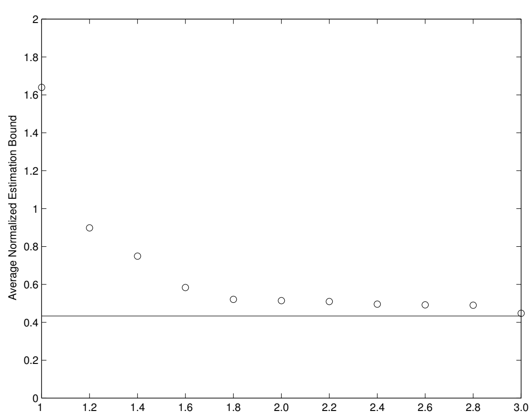

Fig. 6. Circle: CRB using information from local network.
Line: CRB using whole network.

E. CRB under Different Propagation Models

In previous discussion, we always assumed that the ranging information is corrupted by iid Gaussian noises. However from the CRB on the ranging information, Eqn.1, we know that the variance $\sigma_{i,j}^2$ of the additive noise on the distance measurement could depend on the distance $d_{i,j}$ between two nodes i, j, because the received wireless signal $A(t)$ decays as a function of d. In this paper, we assume $\sigma_{i,j}^2 = \sigma^2 d_{i,j}^a$, where σ^2 is the noise variance when $d = 1$.[4] Under the propagation model, we assume the ranging information between any two sensors is available, though it may be quite bad if they are very far apart. Here, we will also ignore the issue of interference among the wireless measurements. This assumption is reasonable only in the case where there is no bandwidth constraint for the system as a whole, or if the data rates of communication are so low that all nodes can use signaling orthogonal to each other.

Define $K = \sigma^2 J$ to be the normalized FIM. Just as in the case where $a = 0$, translations of the whole sensor network do not change the FIM. Rotation does not change the CRB on any node $K^{-1}_{2i-1,2i-1} + K^{-1}_{2i,2i}$. However, zooming does have an effect on the FIM.

Corollary 4: The normalized FIM K is scaled under zooming If the propagation model is $d^a, a \geq 0$,and the whole sensor network is zoomed by a zooming factor $c > 0$. $K(\{c(x_i, y_i)\}) = \frac{1}{c^a} K(\{(x_i, y_i)\})$, $c \neq 0$.

Proof: : Zooming does not change the angles $\alpha_{i,j}$ between sensors, the only thing that changes is the decay factor $d_{i,j}^a$. If the zooming factor is c, then the decaying factor changes to $(cd_{i,j})^a = c^a d_{i,j}^a$, Substitute the new decaying factors into the FIM as in Theorem 1, we get: $K(\{c(x_i, y_i)\}) = \frac{1}{c^a} K(\{(x_i, y_i)\})$. □

The CRB $\sigma^2 K^{-1}_{i,i}$ changes proportional to c^a, if the whole sensor network is zoomed up by a factor c.

Next, we have a simulation where we fix the node density and examine the average CRB for different a's as we vary the size of the sensor network. The sensor network is the same as in Fig.4 and the sizes are taken at $1 \times 1, 3 \times 3, \ldots, 13 \times 13$. We calculate the average CRB inside the central square and plot the aligned average estimation bound in $10 \log_{10}$ scale in Fig.7.

The average CRB decreases as the size of the sensor network increases. This is expected since there is more information available and no interference by assumption. Asymptotically, the CRB decreases at a faster rate for smaller a since the noise variance increases more slowly with range.

Heuristically we have the following explanation for the effects of faraway nodes on the estimation of a single

[4]Earlier, we had a hybrid model with $a = 0$ locally and $a = \infty$ at a great distance since the ranging information is only available for sensor pair i, j, if $d_{i,j} < R_{visible}$.

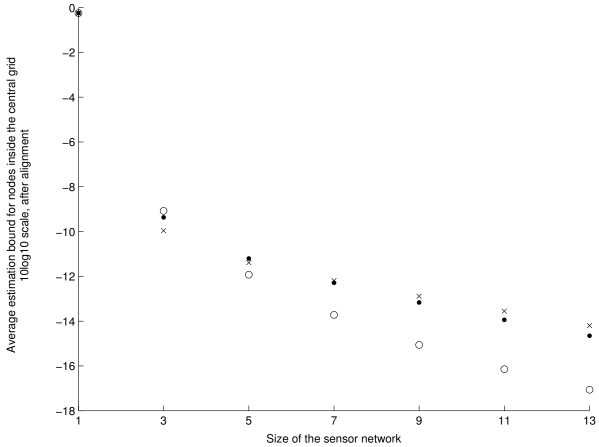

Fig. 7. Average CRB in the central grid for different a
Circle: $a = 1$, Dot: $a = 2$, Cross: $a = 3$

node i. The estimation accuracy of the position of a node i is mainly determined by the total energy received by it. Suppose that the distance between nodes is $\geq r_m$, and the nodes are uniformly distributed. We approximate the total received energy P_R coming from sensors within distance R as:

$$P_R = \beta \int_0^{2\pi} \int_{r_m}^{R} \rho^{-a} \rho d\rho d\theta = 2\beta\pi \int_{r_m}^{R} \rho^{1-a} d\rho$$
$$= \begin{cases} \frac{2\beta\pi}{2-a}(R^{2-a} - r_m^{2-a}) & \text{if } a \neq 2 \\ 2\beta\pi(\ln(R) - \ln(r_m)) & \text{if } a = 2. \end{cases}$$

When $a < 2$, P_R behaves like R^{2-a} which grows unboundedly as the network grows and similarly for $a = 2$ where P_R behaves like $\ln(R)$. In such cases, *there are advantages to having a bigger network and it should be possible to save each node's transmitter power by going to a larger network and then turning down the transmit power in such a way as to keep the position accuracy fixed.* This also suggests that distributed algorithms for positioning in such cases should have some way of using the measurements from distant nodes. When $a > 2$, P_R converges to $\frac{2\beta\pi}{a-2} r_m^{2-a}$. This heuristic explanation is a qualitative fit with Fig.7, when $a = 1$, the difference between P_6 and P_{12} is about $3dB$.

III. Estimation Bounds for Anchor-Free Localization

In the anchor-free localization problem, the only information available is the inter-node distance measurements. The nature of anchor-free localization is very different from anchored localization, in the sense that the absolute positions of the nodes cannot be determined. We first review the singularity of the FIM using the treatment from [17].

Lemma 2: Rank of the FIM: For a n parameter estimation problem, if $\vec{d}$ is the observation vector, and θ is the n dimensional parameter. Write $l(\vec{d}|\theta) = \ln(p(\vec{d}|\theta))$ as the log likelihood function. Then the rank of the FIM J is $n - k$, $k \geq 0$, if and only if the expectation of the square of directional derivative of $l(\vec{d}|\theta)$ at θ is zero for k independent vectors $b_1, \ldots, b_k \in R^n$.

Proof: : The directional derivative of $l(\vec{d}|\theta)$ at θ, along direction b_i is : $\tau(b_i) = (\partial l/\partial\theta_1, \partial l/\partial\theta_2, \ldots, \partial l/\partial\theta_n) b_i$.

$$\begin{aligned} & E(\tau(b_i)^2) \\ & = E(b_i^T (\partial l/\partial\theta_1, \ldots, \partial l/\partial\theta_n)^T (\partial l/\partial\theta_1, \ldots, \partial l/\partial\theta_n) b_i) \\ & = b_i^T J b_i \end{aligned} \quad (22)$$

If k independent vectors $b_1, \ldots, b_k$ make $b_i^T J b_i = 0$, the rank of J is $n - k$, since j is an $n \times n$ symmetric matrix. □

The FIM for anchor-free localization is the same as what was calculated in Theorem 1, just with no anchors. With the above lemma, we can prove that the rank of this FIM is at most $2M - 3$ in a M node sensor network. This is intuitively obvious since there are $2M$ parameters $x_1, y_1, x_2, y_2, \ldots, x_M, y_M$ with 3 degree of freedom in rotation and translation on x_i, y_i.

Theorem 3: For the anchor-free localization problem, with M nodes, the FIM $J(\theta)$ is of rank $2M - 3$

Proof: : The parameter vector $\theta = (x_1, y_1, \ldots, x_M, y_M)$, where The observation vector $\vec{d} = \{\hat{d}_{i,j}, 1 \leq i, j \leq M, j \in adj(i)\}$, where $\hat{d}_{i,j}$ is the measured distance between node i and j, and $\hat{d}_{i,j}$'s are independent given the true distance $\sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}$. Then the log-likelihood function of this estimation problem is :

$$\begin{aligned} & l(\vec{d}|\theta) \\ & = \ln(p(\{\hat{d}_{i,j}, 1 \leq i, j \leq M, j \in adj(i)\} | x_1, y_1, \ldots, x_M, y_M)) \\ & = \ln(p(\{\hat{d}_{i,j}, 1 \leq i, j \leq M, j \in adj(i)\} \\ & |\{\sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}, 1 \leq i, j \leq M, j \in adj(i)\}) \\ & = \sum_{1 \leq i,j \leq M, j \in adj(i)} \ln(p(\hat{d}_{i,j} | \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2})) \end{aligned}$$

The last equality comes from the conditional independence. The directional derivative of each term in the sum is 0 along the vectors $\vec{b}_1, \vec{b}_2, \vec{b}_3 \in R^{2M}$. $\vec{b}_1 = (1, 0, 1, 0, \ldots, 1, 0)^T, \vec{b}_2 = (0, 1, 0, 1, \ldots, 0, 1)^T, \vec{b}_3 = (y_1, -x_1, y_2, -x_2, \ldots, y_M, -x_M)^T$ where $\vec{b}_1$ and $\vec{b}_2$ span the 2-D space in R^{2M} corresponding to translations and

$\vec{b}_3$ is the instantaneous direction when the whole sensor networks rotates. □

Since the rank of the FIM is $2M-3$, we cannot apply the standard CRB argument because J^{-1} does not exist. However, the CRB is $J^{\dagger}$ instead, where $J^{\dagger}$ is the Moore-Penrose pseudo-inverse of J.[20]

A. What does $J^{\dagger}$ mean: total estimation bound

In an n parameter estimation problem where the FIM J is singular, we cannot properly define the parameter estimation problem in R^n. However, we can estimate the parameters in the subspace spanned by all k orthonormal eigenvectors $\vec{v}_1, \ldots, \vec{v}_k$ corresponding non-zero eigenvalues of J. In that subspace, the FIM Q is full rank. Write $V = (v_1, \ldots, v_k)$, V is an $n \times k$ matrix and $V^T V = I_k$, then $Q = V^T J V$, and $Q^{-1} = V^T J^{\dagger} V$, thus $J^{\dagger}$ is the intrinsic CRB matrix for the estimation problem. The total estimation bound for the estimation problem in the k dimensional subspace is $Tr(Q^{-1})$, and $Tr(Q^{-1}) = Tr(J^{\dagger})$ by elementary matrix theory.

Unlike the anchored case, we cannot claim the estimation accuracy of a single node to be bounded by:

$$E((\hat{x_i} - x_i)^2) + E((\hat{y_i} - y_i)^2) \geq J^{\dagger}_{2i-1,2i-1} + J^{\dagger}_{2i,2i} \quad (23)$$

since there always exists a translation of the entire network to make the estimation of node i perfectly accurate. However we find the total estimation bound depicts the estimation performance of anchor-free localization since the trace is invariant.

Definition 1: Total estimation bound $V_{total}(J)$ on anchor-free localization
$V_{total}(J) = \sum_{i=1}^{M} (J^{\dagger}_{2i-1,2i-1} + J^{\dagger}_{2i,2i}) = Tr(J^{\dagger})$
By the definition we know that $V_{total}(K)$ is invariant under rotation, translation and zooming. The total estimation bound is in fact the sum of the inverse of the $2M-3$ non-zero eigenvalues of K.

Theorem 4: Total estimation bound $V_{total}(J)$ on an anchor-free localization problem
$V_{total}(J) = \sum_{i=1}^{2M-3} \frac{1}{\lambda_i}$, where λ_i's are non-zero eigenvalues of J

Proof: The correctness follows the fact that the eigenvalues of $J^{\dagger}$ are $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots, \frac{1}{\lambda_{2M-3}}, 0, 0, 0$. And so $Tr(J^{\dagger}) = \sum_{i=1}^{2M-3} \frac{1}{\lambda_i}$. □

1) Total estimation bound on 3 nodes anchor-free localization: Using Theorem 4, we can give the total lower bound on any geometric setup of an anchor-free localization. The simplest nontrivial case is when there are only 3 points. We fix two points at $(0,0), (0,1)$ and assume the longest edge has length 1. We plot the contour of the total estimation bound as a function of the position of the $3rd$ node $\in [0,1] \times [0,1]$.

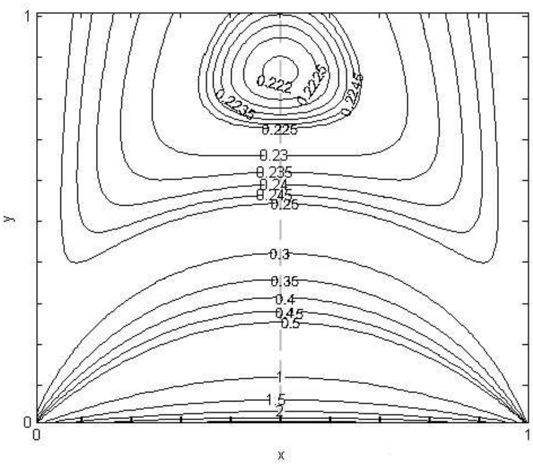

Fig. 8. The contour shows the total estimation bound in $log10$ scale for the 3rd node at (x, y).

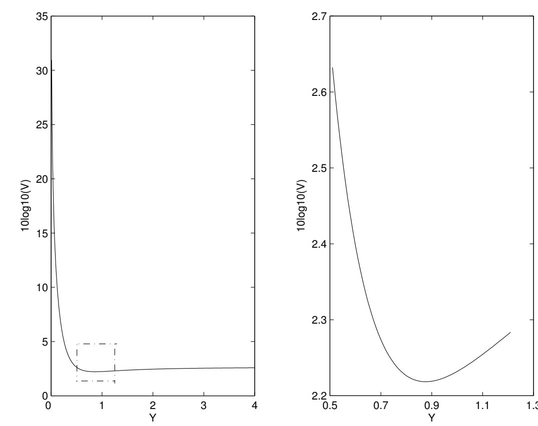

Fig. 9. The total estimation bound, the 3rd node is at $(0.5, y)$ along the gray line in Fig.8.

The result shows that the total estimation bound is related to the biggest angle of the triangle. The larger that angle is, the larger the total estimation bound is. From Fig.8 (b), we find that the minimum total estimation bound is achieved when the triangle is equilateral, where the 3rd node is at $(0.5, \frac{\sqrt{3}}{2})$. In Fig.9 we show what is happening around the minimum.

2) Total estimation bound for different network shapes: The shape of the sensor network effects the total estimation bound. We illustrate this by a simulation with M sensors randomly and uniformly distributed in a region A with all the pairwise distances measured. We plot the average normalized total estimation bound of 50 independent experiments.

In Fig.10 A is a rectangular region with dimension $L_1 \times L_2, L_1 \geq L_2$. Since the zooming does not change the total estimation bound, the only thing matters is the ratio $R = \frac{L_1}{L_2}$, and it turns out that the normalized CRB increases as R increases, or as the rectangular becomes less and less square.[5] However, once the number of nodes had gotten large enough, the total estimation error bound did not change with more nodes. The error was reduced per-node in a way that simply distributed the same total error over a larger number of nodes.

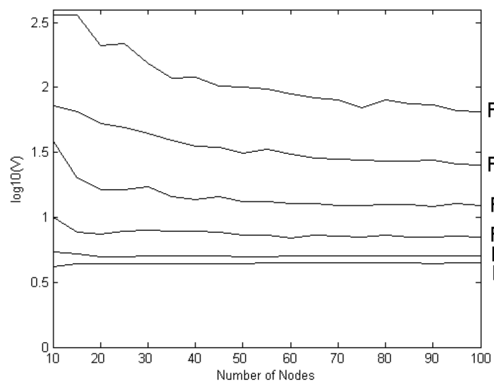

Fig. 10. The normalized total estimation lower bound VS number of nodes. Rectangular region ($R = \frac{L_1}{L_2}$)

B. Why not set a node at $(0, 0)$ and another node on the x axis

It is tempting to eliminate the singularity of the FIM by just setting some parameters. If we fix node 1 at position (x_1, y_1), node 2 with y-coordinate 0, it is equivalent to doing the estimation in the subspace through point $(x_1, y_1, \ldots, x_M, y_M)$ perpendicular to $\vec{c}_1 = (1, 0, 0, 0, \ldots, 0)^T, c_2 = (0, 1, 0, 0, \ldots, 0)^T, \vec{c}_3 = (0, 0, 1, 0, \ldots, 0)^T$. In general, the subspace generated by $\vec{c}_1, \vec{c}_2, \vec{c}_3$ is not the same as that generated by $\vec{b}_1, \vec{b}_2, \vec{b}_3$ and so the choice of which nodes we choose to fix can impact the bounds!

C. Comparison of anchored and anchor-free localization

Sometimes bad geometric setup of anchors results in bad anchored estimation, while the anchor-free estimation is still good! As such, it is not useful to view the anchor-free case as an information-limited version of the anchored case. After all, in the anchored case, we also have a more challenging goal: to get the absolute positions correct, not just up to equivalency. In Fig.11, we have a sensor network with 3 anchors very close to each other, the total estimation bound for anchored localization is 195.20, meanwhile the total estimation bound for anchor-free localization is 4.26.

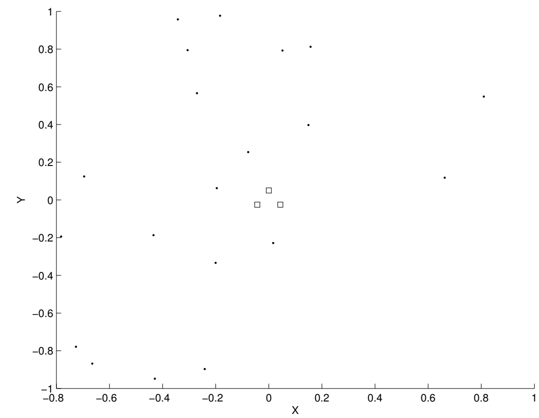

Fig. 11. A bad setup of anchors. square: anchors, dot: nodes

D. Total Estimation Bound under Different Propagation Models

It can be easily seen that just as in the anchored localization, K is invariant under translation and $V_{total}(K)$ is invariant under rotation as well. Just as in anchored localization, the total estimation bound $V_{total}(K)$ changes proportional to c^a, if the whole sensor network is zoomed up by a factor c.

In simulation, we study the affect of the size of the sensor network on the average estimation bound in different propagation models, i.e. for different a's using the same setup as Fig.4.

As shown in Fig.12, we observe that the average estimation bound decreases as the size of the sensor network increases with fixed node density. Just as in the anchored case shown in Fig.7, the estimation accuracy is mainly determined by the received power and so the heuristic explanation for the anchored case also fits into the simulation results we have for the anchor-free case.

IV. Conclusions and Future Work

In this paper, we studied the CRB for both anchored and anchor-free localization and gave a method to compute the CRB in terms of the geometry of the sensor

[5]In [22], we also studied the total estimation bound for an annular region. Let $R = \frac{r_{inner}}{r_{outer}}$ be the ratio of the radius of the inner circle over the radius of the outer circle, we observe that the total estimation bound decreases as R increases and again the total estimation bound is roughly constant with respect to the number of nodes. The best case was having the nodes along the circumference of a circle!

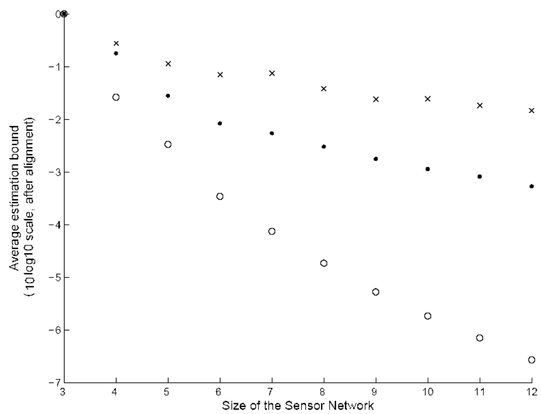

Fig. 12. The average normalized total estimation lower bound VS size of the sensor network for different a.
Circle: $a = 1$, Dot: $a = 2$, Cross: $a = 3$

network. For the anchored localization problem, we derived both lower and upper bounds on the CRB which can be determined by only local geometry. These showed that we can use local geometry to predict the accuracy of the position estimation and that bodes well for distributed algorithms. For the anchor-free localization problem, due to the singularity of the Fisher Information matrix, we computed the total estimation bound instead. Finally, we considered the implications of wireless signal propagations and found that if the signals propagate very well, then there are potential gains by using larger networks and doing estimation in a manner that uses this information. This deserves to be studied in greater detail.

So far, we have only computed CRB on the localization problem. For the design of algorithms, it would also be good to know the sensitivities to individual observations. It might be very helpful to the localization if one can identify the bottleneck of the problem. i.e. how to figure out which distance measurement could help to increase the localization accuracy the most. With the knowledge of the bottleneck, it may be possible to allocate the energy or computation in a smart way to improve localization accuracy.

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