Interference channel capacity region for randomized fixed-composition codes

Cheng Chang
D. E. Shaw & Co, New York, NY.
120 West 45th Street 39th Floor New York, NY 10036
cchang@eecs.berkeley.edu

***Abstract*— The randomized fixed-composition codes with optimal decoding error exponents are recently studied in [11], [12] for the finite alphabet interference channel with two transmitter-receiver pairs. In this paper we investigate the capacity region for randomized fixed-composition codes. A complete characterization of the capacity region of the said coding scheme is given. The inner bound is derived by showing the existence of a positive error exponent within the capacity region. A simple universal decoding rule is given. The tight outer bound is derived by extending a technique first developed in [10] for single input output channels to interference channels. It is shown that even with a sophisticated time-sharing scheme among randomized fixed-composition codes, the capacity region of the randomized fixed-composition coding is not bigger than the known Han-Kobayashi [24] capacity region. This suggests that the study of the average behavior of randomized codes are not sufficient in finding new capacity regions.**

I. Introduction

The interference channel is a channel model with multiple input-output pairs that share a common communication channel [23]. The capacity region, within which reliable communication can be achieved for *all* input-output pairs, has been studied [23], [1], [3], [2]. The most well known capacity region result is given in [24], where the capacity region is studied for both discrete and Gaussian cases. Some recent progresses on the capacity region are reported in [14], [19], [22], [5], [13]. However, the capacity regions for general interference channels are still unknown. We focus our investigation on the capacity region for a specific coding scheme: randomized fixed-composition codes for which the error probability is defined as the average error probability over all code books with a certain composition (type). Fixed-composition coding is a useful coding scheme in the investigation of both upper [15] and lower bounds of channel coding error exponents [8] for point to point channel and [21], [20] for multiple access (MAC) channels. Recently in [11] and [12], randomized fixed-composition codes were used to derive a lower bound on the error exponent for discrete memoryless interference channels. A lower bound on the maximum-likelihood decoding error exponent is derived, this is a new attempt in investigating the error exponents for interference channels. The unanswered question is the capacity region of such coding schemes.

This work was done when he was a postdoctoral researcher with the Hewlett-Packard Laboratories, Palo Alto, CA.

We answer the above question by giving a complete characterization of the interference channel capacity region for randomized fixed-composition codes. To prove the achievability of the capacity region, we prove the positivity of an achievable error exponent everywhere inside the capacity region. This error exponent is derived by using the method of types [7], in particular, the universal decoding scheme used for multiple-access channels [21]. A better error exponent can be achieved by using the more complicated universal decoding rules developed in [20]. But since they both have the same achievable capacity region, we use the simpler scheme in [21]. To prove the converse, that the achievable region matches the outer bound, we extend the technique in [10] for point to point channels to interference channels by using the known capacity region results for multiple-access channels. The result reveals the intimate relations between interference channels and multiple-access channels. With the capacity region for fixed-composition code established, it is evident that this capacity region is a subset of the Han-Kobayashi region [24].

In this paper we focus on the two input-output case and study the discrete memoryless interference channels with transition probability $W_{Z|X,Y}$ and $\tilde{W}_{\tilde{Z}|X,Y}$ respectively as shown in Figure 1. The two channel inputs are $x^n \in \mathcal{X}^n$ and $y^n \in \mathcal{Y}^n$, outputs are $z^n \in \mathcal{Z}^n$ and $\tilde{z}^n \in \tilde{\mathcal{Z}}^n$ respectively, where $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$ and $\tilde{\mathcal{Z}}$ are finite sets. We study the basic interference channel where each encoder only has a private message to the corresponding decoder.

The technical proof of this paper is focused on the average behavior of fixed-composition code books. However this fundamental setup can be extended in the following three directions.

- It is obvious that there exists a code book that its decoding error is no bigger than the average decoding error over all code books. Hence the achievability results in this paper guarantees the existence of a of deterministic coding scheme with at least the same error exponents and capacity region. More discussions are in Section II-E.
- The focus of this paper is on the fixed-composition codes with a composition P, where P is a distribution on the input alphabet. This code book generation is different from the non-fixed-composition random coding [16] according to distribution P. It is well known in the literature that the fixed-composition codes gives better error exponent result in low rate regime for point to point channels [8]

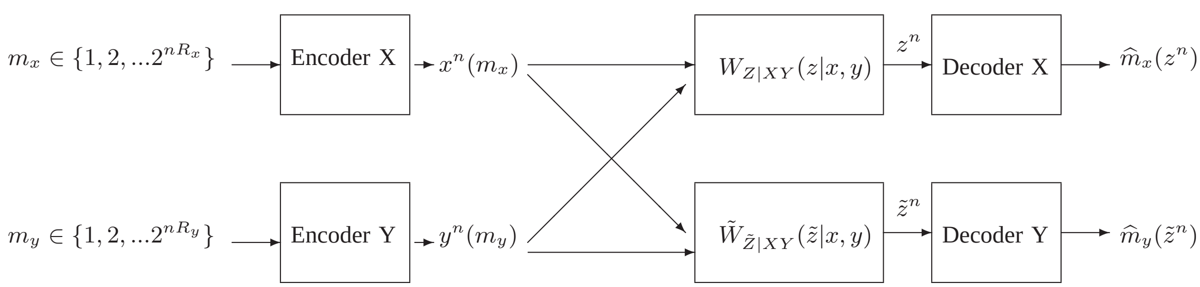

Fig. 1. A discrete memoryless interference channel of two users

and multiple-access channels [21], [20]. However they have the same achievable rate region. It is the same case for interference channels and hence the capacity region result in this paper applies to the non-fixed-composition random codes.

- Time-sharing is a key element in achieving capacity regions for multi-terminal channels [6]. For instance, for multiple-access channels, simple time-sharing among fixed-composition codes gives the entire capacity region. We show that our fixed composition codes can be used to build a time-sharing capacity region for the interference channel. More interestingly, we show that the simple time-sharing technique that gives the entire capacity region for multiple-access channels is not enough to get the largest capacity region, a more sophisticated time-sharing scheme is needed. Detailed discussions are in Section IV.

The outline of the paper is as follows. In Section II we first formally define randomized fixed-composition codes and its capacity region and then in Section II-C we present the main result of this paper: the interference channel capacity region for randomized fixed-composition codes in Theorem 1. The proof is later briefly explained in Section III with more details in [4]. Finally in Section IV, we argue that due to the non-convexity of capacity region of the randomized fixed-composition codes. A more sophisticated time-sharing scheme is needed. This shows the necessity of studying the geometry[1] of the code-books for interference channels.

II. Randomized fixed-composition codes and its capacity region

We first review the definition of randomized fixed-composition code that is studied intensively in previous works [9], [10], [21], [20]. Then the definition of the interference channel capacity region for such codes is introduced. Then we give the main result of this paper: the complete characterization of the capacity region for randomized fixed-composition codes.

A. Randomized fixed-composition codes

A randomized fixed-composition coding system is a uniform distribution on the code books in which every codeword is from the type set with the fixed composition (type).

First we introduce the notion of type set [6]. A type set $\mathcal{T}^n(P)$ is a set of all the strings $x^n \in \mathcal{X}^n$ with the same type P where P is a probability distribution [6]. A sequence of type sets $\mathcal{T}^n \subseteq \mathcal{X}^n$ has composition P_X if the types of $\mathcal{T}^n$ converges to P_X, i.e. $\lim_{n\to\infty} \frac{N(a|\mathcal{T}^n)}{n} = P_X(a)$ for all $a \in \mathcal{X}$ that $P_X(a) > 0$ and $N(a|\mathcal{T}^n) = 0$ for all $a \in \mathcal{X}$ that $P_X(a) = 0$, where $N(a|\mathcal{T}^n)$ is the number of occurrence of a in type $\mathcal{T}^n$. We ignore the nuisance of the integer effect and assume that $nP_X(a)$ is an integer for all $a \in \mathcal{X}$ and nR_x and nR_y are also integers. This is indeed a reasonable assumption since we study long block length n and all the information theoretic quantities studied in this paper are continuous on the code compositions and rates. We simply denote by $\mathcal{T}^n(P_X)$ the length-n type set which has "asymptotic" type P_X.Allowing identical codewords for difference messages in the same code book, there are $|\mathcal{T}^n(P_X)|^{2^{nR_x}}$ many code books with fixed-composition P_X and rate R_x.

In this paper, we study the randomized fixed-composition codes, where each code book with all codewords from the fixed composition being chosen with the same probability. Equivalently, over all these code books, a codeword for message i is uniformly i.i.d distributed on the type set $\mathcal{T}^n(P_X)$. A formal definition is as follows.

Definition 1: Randomized fixed-composition codes: for a probability distribution P_X on $\mathcal{X}$, a rate R_x randomized fixed-composition-P_X encoder picks a code book with the as follows. For any length-n fixed-composition-P_X code book $c_X = (x^n(1), x^n(2), ..., x(2^{nR_x}))$, where $x^n(i) \in \mathcal{T}^n(P_X)$, $i = 1, 2, ..., 2^{nR_x}$, and $x^n(i)$ and $x^n(j)$ may not be different

[1]A code book of rate R and of code length n can be viewed as a set of 2^{nR} points located in the codeword space $\mathcal{X}^n$, hence the geometry of a code book is the relations among these 2^{nR} points.

for $i \neq j$, the code book c_X is chosen with probability

$$\left(\frac{1}{|\mathcal{T}^n(P_X)|}\right)^{2^{nR_x}}.$$

In other words, the choice of the code book is a random variable that is uniformly distributed on the index set of all code books with fixed-composition P_X: $\{1, 2, 3, ..., |\mathcal{T}^n(P_X)|^{2^{nR_x}}\}$. The chosen code book c_X is shared between the encoder X and the decoders X and Y.

The key property of the randomized fixed-composition codes is that for any message subset $\{i_1, i_2, ...i_l\} \subseteq \{1, 2, ..., 2^{nR_x}\}$, the codewords for these messages are identical independently distributed on the type set $\mathcal{T}^n(P_X)$.

For randomized fixed-composition codes, the average error probability $P^n_{e(x)}(R_x, R_y, P_X, P_Y)$ for X is the expectation of decoding error over all message, code books and channel behaviors [2].

$$P^n_{e(x)}(R_x, R_y, P_X, P_Y) = \left(\frac{1}{|\mathcal{T}^n(P_X)|}\right)^{2^{nR_x}} \quad (1)$$
$$\left(\frac{1}{|\mathcal{T}^n(P_Y)|}\right)^{2^{nR_y}} \sum_{c_X}\sum_{c_Y}\frac{1}{2^{nR_x}}\sum_{m_x}\frac{1}{2^{nR_y}}\sum_{m_y}\sum_{z^n}$$
$$W_{Z|XY}(z^n|x^n(m_x), y^n(m_y))1(\widehat{m}_x(z^n) \neq m_x)$$

where $x^n(m_x)$ is the codeword of message m_x in code book c_X, similarly for $y^n(m_y)$, $\widehat{m}_x(z^n)$ is the decision made by the decoder knowing the code books c_X and c_Y.

B. *Randomized fixed-composition coding capacity for interference channels*

Given the definitions of randomized fixed-composition coding and the average error probability in (1) for such codes, we can formally define the capacity region for such codes.

Definition 2: Capacity region for randomized fixed-composition codes: for a fixed-composition P_X and P_Y, a rate pair (R_x, R_y) is said to be achievable for X, if for all $\delta > 0$, there exists $N_\delta < \infty$, s.t. for all $n > N_\delta$,

$$P^n_{e(x)}(R_x, R_y, P_X, P_Y) < \delta \quad (2)$$

We denote by $\mathcal{R}_x(P_X, P_Y)$ the closure of the union of the all achievable rate pairs. Similarly we denote by $\mathcal{R}_y(P_X, P_Y)$ the achievable region for Y, and $\mathcal{R}_{xy}(P_X, P_Y)$ for (X, Y) where both decoding errors are small. Obviously

$$\mathcal{R}_{xy}(P_X, P_Y) = \mathcal{R}_x(P_X, P_Y)\bigcap \mathcal{R}_y(P_X, P_Y). \quad (3)$$

We only need to focus our investigation on $\mathcal{R}_x(P_X, P_Y)$, then by the obvious symmetry, both $\mathcal{R}_y(P_X, P_Y)$ and $\mathcal{R}_{xy}(P_X, P_Y)$ follow.

[2]To simplify notations, we denote by $\sum_{c_X}$ the sum over all composition P_X and rate R_x code book c_X in (1).

C. *Capacity region of the fixed-composition code, $\mathcal{R}_x(P_X, P_Y)$, for X*

The main result of this paper is the complete characterization of the randomized fixed-composition capacity region $\mathcal{R}_x(P_X, P_Y)$ for X, as illustrated in Figure 2. By symmetry, $\mathcal{R}_y(P_X, P_Y)$ and then $\mathcal{R}_{xy}(P_X, P_Y)$ follow.

Theorem 1: Interference channel capacity region $\mathcal{R}_x(P_X, P_Y)$ for randomized fixed-composition codes with compositions P_X and P_Y:

$$\begin{aligned}
&\mathcal{R}_x(P_X, P_Y) = \\
&\{(R_x, R_y) : 0 \leq R_x < I(X; Z), 0 \leq R_y\}\bigcup \\
&\{(R_x, R_y) : 0 \leq R_x < I(X; Z|Y), \\
&\quad R_x + R_y < I(X, Y; Z)\}
\end{aligned} \quad (4)$$

where the random variables in (4), $(X, Y, Z) \sim P_X P_Y W_{Z|X,Y}$. The region $\mathcal{R}_x(P_X, P_Y)$ is illustrated in Figure 2.

The achievable part of the theorem states that: for a rate pair $(R_x, R_y) \in \mathcal{R}_x(P_X, P_Y)$, the union of Region I and II in Figure 2, for all $\delta > 0$, there exists $N_\delta < \infty$, s.t. for all $n > N_\delta$, the average error probability (1) for the randomized code from compositions P_X and P_Y is smaller than δ for X:

$$P^n_{e(x)}(R_x, R_y, P_X, P_Y) < \delta$$

for some decoding rule. Region II is also the multiple-access capacity region for fixed-composition codes (P_X, P_Y) for channel $W_{Z|XY}$.

The converse of the theorem states that for any rate pair (R_x, R_y) outside of $\mathcal{R}_x(P_X, P_Y)$, that is region III, IV and V in Figure 2, there exists $\delta > 0$, such that for **all** n,

$$P^n_{e(x)}(R_x, R_y, P_X, P_Y) > \delta$$

no matter what decoding rule is used. The definition of the error probability $P^n_{e(x)}(R_x, R_y, P_X, P_Y)$ is average over all code books and channel realizations as defined in (1).

The sketch of the proof of Theorem 1 is in Section III with details in [4].

There are two important observations here. First, the capacity region achieved for x and y defined as $\mathcal{R}_{xy}(P_X, P_Y) = \mathcal{R}_x(P_X, P_Y)\bigcap \mathcal{R}_y(P_X, P_Y)$ is a subset of the capacity region proposed by Han and Kobayashi in [24] which is convex. Hence the randomized fixed-composition codes do not give a bigger capacity region than that in [24]. Secondly, the converse of the randomized coding does not guarantee that there is not a single good fixed-composition code book. The converse claims that, the average (over all code books with the fixed composition) decoding error probability does not converge to zero if the rate pair is outside the capacity region in Theorem 1.

The significance of the above two points is that we cannot hope to get a bigger capacity region than that in [24] by using the fixed-composition random coding scheme which is

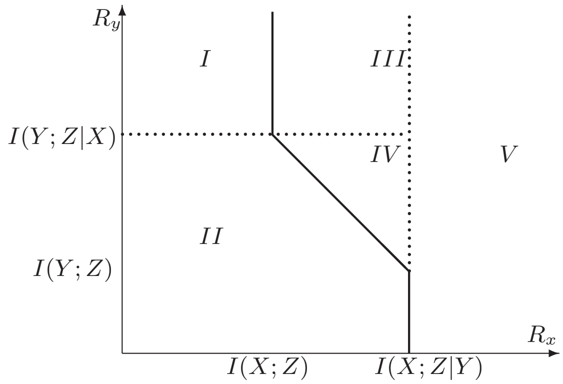

Fig. 2. Randomized fixed-composition capacity region $\mathcal{R}_x(P_X,P_Y)$ for X, the achievable region is the union of Region I and II.

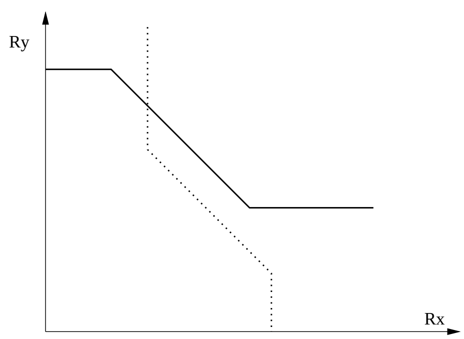

Fig. 3. A typical randomized fixed-composition capacity region $\mathcal{R}_{xy}(P_X,P_Y) = \mathcal{R}_x(P_X,P_Y) \cap \mathcal{R}_y(P_X,P_Y)$ is the intersection of the dotted line and the solid lines, this capacity region is not necessarily convex.

sufficient to achieve the entire capacity regions for point to point channel [9], multiple access channels [18] and degraded broadcast channels [17]. This confirms that interference channel coding is a more difficult problem.

D. Necessity of more sophisticated time-sharing schemes

In the achievability part of Theorem 1, we prove that the average error probability for X is arbitrarily small for randomized fixed-composition codes if the rate pair (R_x,R_y) is inside the capacity region $\mathcal{R}_x(P_X,P_Y)$. For interference channels, it is obvious that the rate region for both X and Y is:

$$\mathcal{R}_{xy}(P_X,P_Y) = \mathcal{R}_x(P_X,P_Y) \cap \mathcal{R}_y(P_X,P_Y), \tag{5}$$

where $\mathcal{R}_y(P_X,P_Y)$ is defined in the same manner as $\mathcal{R}_x(P_X,P_Y)$ but the channel is $\tilde{W}_{\tilde{Z}|XY}$ instead of $W_{Z|XY}$ as shown in Figure 1. A typical capacity region $\mathcal{R}_{xy}(P_X,P_Y)$ is shown in Figure 3. It is not necessarily convex.

However, by a simple time-sharing between different rate pairs for the same composition, we can convexify the capacity region. Then the convex hull of the union of all such capacity regions of different compositions gives a bigger convex achievable capacity region. This capacity region of the interference channel is

$$CONVEX\left(\bigcup_{P_X,P_Y} \mathcal{R}_{xy}(P_X,P_Y)\right).$$

It is tempting to claim that the above convex capacity region is the largest one can get by time-sharing the "basic" fixed-composition codes as multiple-access channels shown in [6]. However, as will be discussed later in Section IV, it is not the case. A more sophisticated time-sharing gives a bigger capacity region.

This is an important difference between interference channel coding and multiple-access channel coding because the fixed-composition capacity region is convex for the latter and hence the simple time-sharing gives the biggest(entire) capacity region [6]. Time-sharing capacity is detailed in Section IV.

E. Existence of a good code for an interference channel

In this paper we focus our study on the average (over all messages) error probability over all code books with the same composition. For a rate pair (R_x,R_y), if the average error probability for X is smaller than δ, then obviously there exists a code book such that the error probability is smaller than δ for X. This should be clear from the definition of error probability $P^n_{e(x)}(R_x,R_y,P_X,P_Y)$ in (1). In the following example, we illustrate that this is also the case for decoding error for both X and Y. We claim without proof that this is also true for "uniform" time-sharing coding schemes later discussed in Section IV. The existence of a code book pair that achieves the error exponents in the achievability part of the proof of Theorem 1 can also be shown. The proof is similar to that in [16] and Exercise 30 (b) on page 198 [9].

Similar to the error probability for X defined in (1), we define the average joint error probability for X and Y as

$$
\begin{aligned}
&P^n_{e(xy)}(R_x, R_y, P_X, P_Y) = \\
&\quad \left(\frac{1}{|\mathcal{T}^n(P_X)|}\right)^{2^{nR_x}} \left(\frac{1}{|\mathcal{T}^n(P_Y)|}\right)^{2^{nR_y}} \\
&\quad \sum_{c_X}\sum_{c_Y}\frac{1}{2^{nR_x}}\sum_{m_x}\frac{1}{2^{nR_y}}\sum_{m_y} \\
&\qquad \Big\{\sum_{z^n} W_{Z|XY}(z^n|x^n(m_x), y^n(m_y)) 1(\widehat{m}_x(z^n) \neq m_x) \\
&\quad + \sum_{\tilde{z}^n} \tilde{W}_{\tilde{Z}|XY}(\tilde{z}^n|x^n(m_x), y^n(m_y)) 1(\widehat{m}_y(\tilde{z}^n) \neq m_y)\Big\}
\end{aligned} \tag{6}
$$

For a rate pair $(R_x, R_y) \in \mathcal{R}_{xy}(P_X, P_Y) = \mathcal{R}_x(P_X, P_Y) \bigcap \mathcal{R}_y(P_X, P_Y)$, we know that for all $\delta > 0$, there exists $N_\delta < \infty$, s.t. for all $n > N_\delta$, the average error probability is smaller than δ for user X and user Y: $P^n_{e(x)}(R_x, R_y, P_X, P_Y) < \delta$ and $P^n_{e(y)}(R_x, R_y, P_X, P_Y) < \delta$. It is easy to see that the average joint error probability for user X and Y can be bounded by:

$$
\begin{aligned}
& P^n_{e(xy)}(R_x, R_y, P_X, P_Y) \\
= \;& P^n_{e(x)}(R_x, R_y, P_X, P_Y) + P^n_{e(y)}(R_x, R_y, P_X, P_Y) \\
\leq \;& 2\delta
\end{aligned} \tag{7}
$$

From (6), we know that $P^n_{e(xy)}(R_x, R_y, P_X, P_Y)$ is the average error probability of *all* (P_X, P_Y)-fixed-composition codes. Together with (7), we know that there exists at least *one* code book pair such that the error probability is no bigger than 2δ.

III. Sketch of the proof of Theorem 1

There are two parts of the theorem, achievability and converse. The achievability part is proved by showing that by using a maximum mutual information decoding rule, positive error exponents exist everywhere in the capacity region in Theorem 1. We apply a method of types argument that is well known for randomized fixed-composition code in the point to point channel coding [9] and MAC channel coding [21]. The converse is proved by giving a non-vanishing lower bound on the error probability outside the capacity region defined in Theorem 1. In the proof, we extended the technique first developed in [10] for point to point channels to interference channels. Due to the page limit, we ignore the proof here. Details are in [4].

IV. Discussions on Time-sharing

The main result of this paper is the randomized fixed-composition coding capacity region for X that is $\mathcal{R}_x(P_X, P_Y)$ shown in Figure 2. So obviously, the interference channel capacity region, where decoding errors for both X and Y are small, is the intersection of $\mathcal{R}_x(P_X, P_Y)$ and $\mathcal{R}_y(P_X, P_Y)$ where $\mathcal{R}_y(P_X, P_Y)$ is defined in the similar way but with channel $\tilde{W}_{\tilde{Z}|XY}$ instead of $W_{Z|XY}$. The intersected region defined in (5), $\mathcal{R}_{xy}(P_X, P_Y)$, is in general non-convex as shown in Figure 3. Similar to multiple-access channels capacity region, studied in Chapter 15.3 [6], we use this capacity region $\mathcal{R}_{xy}(P_X, P_Y)$ as the building blocks to generate larger capacity regions.

A. A digression to MAC channel capacity region

Before giving the time-sharing results for interference channels and show why the simple time-sharing idea works for MAC channels but not for interference channels, we first look at $\mathcal{R}_x(P_X, P_Y)$ in Figure 2. Region II is obviously the multiple access channel $W_{Z|XY}$ region achieved by input composition (P_X, P_Y) at the two encoders, denoted by $\mathcal{R}^{mac}_{xy}(P_X \times P_Y)$. In [6], the full description of the MAC channel capacity region is given in two different manners:

$$
\begin{aligned}
& CONVEX\left(\bigcup_{P_X, P_Y} \mathcal{R}^{mac}_{xy}(P_X \times P_Y)\right) \\
= \;& CLOSURE \\
& \left(\bigcup_{P_U, P_{X|U}, P_{Y|U}} \mathcal{R}^{mac}_{xy}(P_{X|U} \times P_{Y|U} \times P_U)\right)
\end{aligned} \tag{8}
$$

where $R^{mac}_{xy}(P_{X|U} \times P_{Y|U} \times P_U) = \{(R_x, R_y) : R_x \leq I(X;Z|Y,U), R_y \leq I(Y;Z|X,U), R_x + R_y \leq I(X,Y;Z|U)\}$ and U is the time-sharing auxiliary random variable and $|\mathcal{U}| = 4$.

The LHS of (8) is the convex hull of all the fixed-composition MAC channel capacity regions. The RHS of (8) is the closure (without convexification) of all the time-sharing MAC capacity regions.The equivalence in (8) is non-trivial, it is not a consequence of the tightness of the achievable region. It hinges on the convexity of the "basic" capacity regions $\mathcal{R}^{mac}_{xy}(P_X, P_Y)$. As will be shown in Section IV-C, this is not the case for interference channels, i.e. (8) does not hold anymore.

B. Simple time-sharing capacity region and error exponent

The simple idea of time-sharing is well studied for multi-user channel coding, broadcast channel coding. Whenever there are two operational points $(R^1_x, R^1_y), (R^2_x, R^2_y)$, while there exist two coding schemes to achieve small error probability at each operational point, one can use λn amount of channel uses at (R^1_x, R^1_y) with coding scheme 1 and $(1-\lambda)n$ amount of channel uses at (R^2_x, R^2_y) with coding scheme 2. The rate of this coding scheme is $(\alpha R^1_x + (1-\alpha)R^2_x, \alpha R^1_y + (1-\alpha)R^2_y)$ and the error probability is still small[3] (no bigger than the sum of two small error probabilities). This idea is easily generalized to more than 2 operational points.

This simple time sharing idea works perfectly for MAC channel coding as shown in (8). The whole capacity region can be described as time sharing among fixed-composition codes where the fixed-composition codes are building blocks. If we extend this idea to interference channel, we have the following simple time sharing region as discussed in Section II-D:

[3] The error exponent is, however, at most half of the individual error exponent.

$$CONVEX\left(\bigcup_{P_X,P_Y}\mathcal{R}_{xy}(P_X,P_Y)\right)= \tag{9}$$
$$CONVEX\left(\bigcup_{P_X,P_Y}\mathcal{R}_x(P_X,P_Y)\bigcap R_y(P_X,P_Y)\right).$$

We shall soon see in the next section that this result can be improved.

C. Beyond simple time-sharing: "Uniform" time-sharing

In this section we give a time-sharing coding scheme that was first developed by Gallager [18] and later further studied for universal decoding by Pokorny and Wallmeier [21] to get better error exponents for MAC channels. This type of "uniform" time-sharing schemes not only achieves better error exponents, more importantly, we show that it achieves **bigger** capacity region than the simple time-sharing scheme does for interference channels! Unlike the multiple-access channels where the simple time-sharing achieves the whole capacity region, this is unique to the interference channels, due to the fact that the capacity region is the convex hull of the intersections of pairs of non-convex regions (convex or not is not the issue here, the real difference is the intersection operation).

The organization of the discussions parallels that for that of the fixed-composition coding. We first introduce the "uniform" time-sharing coding scheme, then give the achievable error exponents and lastly drive the achievable rate region for such coding schemes. The proofs are omitted since they are similar to those for the randomized fixed-composition codes.

Definition 3: "Uniform" time-sharing codes: for a probability distribution P_U on $\mathcal{U}$, where $\mathcal{U}=\{u_1,u_2,...,u_K\}$ with $\sum_{i=1}^K P_U(u_i)=1$, and a pair of conditional independent distributions $P_{X|U},P_{Y|U}$. We define the two codeword sets[4] as

$$X_c(n)=\{x^n : x_1^{nP_U(u_1)}\in P_{X|u_1}, x_{nP_U(u_1)+1}^{n(P_U(u_1)+P_U(u_2))}\in P_{X|u_2},...,x_{n(1-P_U(u_K))+1}^{n}\in P_{X|u_K}\}$$

i.e. the i'th chunk of the codeword x^n with length $nP_U(u_i)$ has composition $P_{X|u_i}$, and similarly

$$Y_c(n)=\{y^n : y_1^{nP_U(u_1)}\in P_{Y|u_1}, y_{nP_U(u_1)+1}^{n(P_U(u_1)+P_U(u_2))}\in P_{Y|u_2},...,y_{n(1-P_U(u_K))+1}^{n}\in P_{Y|u_K}\}.$$

A "uniform" time-sharing code $(R_x,R_y,P_UP_{X|U}P_{Y|U})$ encoder picks a code book with the following probability: for any message $m_x\in\{1,2,...,2^{nR_x}\}$, the codeword $x^n(m_x)$ is uniformly distributed in $X_c(n)$, similarly for encoder Y.

After the code book is randomly generated and revealed to the decoder, the decoder uses a maximum mutual information decoding rule. Similar to the fixed-composition coding, the decoder needs to either decode both messages X and Y jointly or simply treats Y as noise and decode X only, depending on where the rate pairs are in Region I or II, as shown in Figure 4. The error probability we investigate is again the average error probability over all messages and code books.

Theorem 2: Interference channel capacity region $\mathcal{R}_x(P_UP_{X|U}P_{Y|U})$ for "uniform" time-sharing codes with composition $P_UP_{X|U}P_{Y|U}$:

$$\begin{aligned}\mathcal{R}_x(P_UP_{X|U}P_{Y|U}) =\\ &\{(R_x,R_y): 0\le R_x< I(X;Z|U), 0\le R_y\}\quad\bigcup\\ &\{(R_x,R_y): 0\le R_x< I(X;Z|Y,U),\\ &\quad R_x+R_y< I(X,Y;Z|U)\}\end{aligned} \tag{10}$$

where the random variables in (10), $(U,X,Y,Z)\sim P_UP_{X|U}P_{Y|U}W_{Z|X,Y}$. And the interference capacity region for $P_UP_{X|U}P_{Y|U}$ is

$$\begin{aligned}&\mathcal{R}_{xy}(P_UP_{X|U}P_{Y|U})=\\ &\quad\mathcal{R}_x(P_UP_{X|U}P_{Y|U})\bigcap\mathcal{R}_y(P_UP_{X|U}P_{Y|U})\end{aligned} \tag{11}$$

The cardinality of $\mathcal{U}$ is shown to be no bigger than 7 by using the Carath$\acute{e}$odory Theorem similar to that in [6] for the capacity region for multiple access channels. We ignore the proof here.

The rate region defined in (10) itself does not give any new X-capacity regions for X, since both Region I and II in Figure 4 can be achieved by simple time-sharing of Region I and II repectively in (4). But for the interference channel capacity, we argue in the next section that this coding scheme gives a strictly bigger capacity region than that given by the simple time-sharing of fixed-composition codes in (9).

The proof of Theorem 2 is similar to that of Theorem 1. Details are in [4].

D. Why the "uniform" time sharing is needed?

It is well understood in the literature [18], also briefly discussed in Section IV-B, that the "uniform" time-sharing fixed-composition coding gives a bigger error exponent than the simple time-sharing coding does. More interestingly, we argue that it gives a bigger interference channel capacity region. First we write down the interference channel capacity region generated from the basic "uniform" time-sharing fixed-composition codes:

$$CONVEX\left(\bigcup_{P_{X|U}P_{Y|U}P_U}\mathcal{R}_{xy}(P_UP_{X|U}P_{Y|U})\right) \tag{12}$$

where $\mathcal{R}_{xy}(P_UP_{X|U}P_{Y|U})$ is defined in (11) and $CONVEX(A)$ is the convex hull (simple time sharing) of set A.

U is a time-sharing auxiliary random variable. Unlike the MAC coding problem, where simple time-sharing of fixed-composition codes achieve the full capacity region, it is not

[4]Again, we ignore the nuisance of the non-integers here.

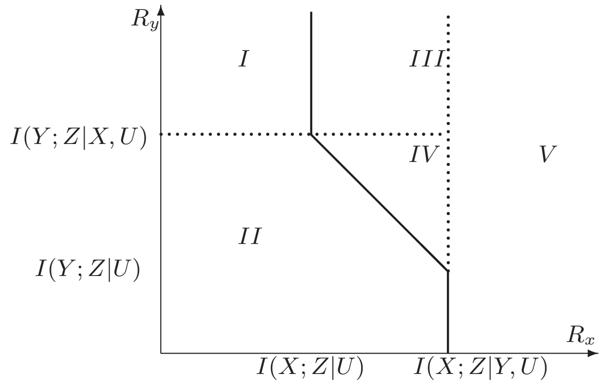

Fig. 4. "Uniform" time-sharing capacity region $\mathcal{R}_x(P_U P_{X|U} P_{Y_U})$ for X, the achievable region is the union of Region I and II. This region is very similar to that for fixed-composition coding shown in Figure 2, only difference is now there is an auxiliary time-sharing random variable U.

guaranteed for interference channels. The reason is the intersection operator in the basic building blocks in (5) and (11) respectively, i.e. the interference nature of the problem[5].

Obviously the rate region by simple time sharing of fixed composition codes in (9) is a subset of simple time sharing of the "uniform" time sharing capacity region (12). In the following example, we illustrate why (12) is bigger than (9).

Example (symmetric interference channels): Suppose that we have a symmetric interference channel, i.e. the input alphabet sets $\mathcal{X} = \mathcal{Y}$ and output alphabet sets $\mathcal{Z} = \tilde{\mathcal{Z}}$ and for any $x, y \in \mathcal{X} = \mathcal{Y}$ and for any $z \in \mathcal{Z} = \tilde{\mathcal{Z}}$: the transition probabilities $W_{Z|XY}(z|x,y) = \tilde{W}_{\tilde{Z}|XY}(z|y,x)$. For such interference channels, we know that the capacity regions $\mathcal{R}_x(P_X, P_Y)$ and $\mathcal{R}_y(P_Y, P_X)$ are symmetric along the 45-degree line $R_x = R_y$. That is, for any P_X, P_Y, a rate pair $(R_1, R_2) \in \mathcal{R}_x(P_X, P_Y)$ if and only if $(R_2, R_1) \in \mathcal{R}_y(P_Y, P_X)$.

The comparison of simple timesharing capacity region and the more sophisticated time-sharing fixed-composition capacity region for symmetric interference channels are illustrated by a toy example in Figure 5. For a distribution (P_X, P_Y), the achievable region for the fixed-composition codes is illustrated in Figure 5, $\mathcal{R}_x(P_X, P_Y)$ and $\mathcal{R}_y(P_X, P_Y)$ respectively, these are bounded by the red dotted lines and red dash-dotted lines respectively, so the interference capacity region $\mathcal{R}_{xy}(P_X, P_Y)$ is bounded by the pentagon $ABEFO$. By symmetry, $\mathcal{R}_x(P_Y, P_X)$ and $\mathcal{R}_y(P_X, P_Y)$ are bounded by the blue dotted lines and blue dash-dotted lines respectively, the capacity region $\mathcal{R}_x y(P_Y, P_X)$ is bounded by the pentagon $HGCDO$. So the convex hull of these two regions is

[5]To understand why intersection is the difference but not the non-convexity, we consider the following toy example: four convex sets: A_1, A_2, B_1, B_2. We show that $CONVEX(A_1 \bigcap B_1, A_2 \bigcap B_2)$ can be strictly smaller than $CONVEX(A_1, A_2) \bigcap CONVEX(B_1, B_2)$. Let $A_1 = B_2 \subset B_1 = A_2$, then $CONVEX(A_1 \bigcap B_1, A_2 \bigcap B_2) = A_1$ is strictly smaller than $CONVEX(A_1, A_2) \bigcap CONVEX(B_1, B_2) = A_2$.

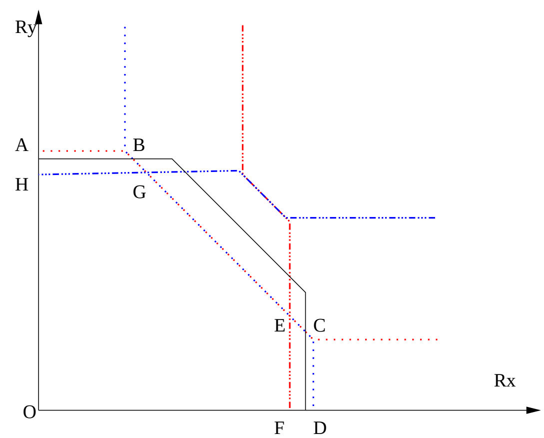

Fig. 5. Simple timesharing of fixed-composition capacity $ABCDO$ VS time-sharing fixed composition capacity(0.5) (the black pentagon)

$ABCDO$.

Now consider the following timesharing fixed-composition coding $P_{X|U} P_{Y|U} P_U$ where $\mathcal{U} = \{0, 1\}$, $P_U(0) = P_U(1) = 0.5$ and $P_{X|0} = P_{Y|1} = P_X$, $P_{X|1} = P_{Y|0} = P_Y$. The interference capacity region is obviously bounded by the black pentagon in Figure 5. This toy example shows why (12) is bigger than (9).

V. Future directions

The most interesting issue of interference channels is the geometry of the two code books. For point to point channel coding, the codewords in the optimal code book is uniformly distributed on a sphere of the optimal compositions and the optimal composition achieves the capacity. For multiple access channels, a simple time-sharing among different fixed-composition codes is sufficient to achieve the whole capacity region, where for each fixed-composition codes, the codewords are uniformly distributed. To get the biggest possible achievable rate region for interference channels, however, as illustrated in Section IV, a more sophisticated "uniform" time sharing is needed. So what is time sharing? Both simple time sharing and "uniform" time sharing change the geometry of the code books, however, in different ways. Simple time sharing "glue" segments of codewords together due to the independence of the coding in different segments of the channel uses, meanwhile for "uniform" time sharing, codewords still have equal distances between one another. Better understanding of the geometry of code books will help us better understand the interference channels. In this paper, we give a tight outer bound to a class of coding schemes, the time-sharing fixed-composition code. An important future direction is to categorize the coding schemes for interference channels and more outer bound result may follow. This is in contrast to the traditional outer bound derivations [3] where genies are used.

ACKNOWLEDGMENTS

The author thanks Raul Etkin, Neri Merhav and Erik Ordentlich for introducing the problem and helpful discussions along the way.

REFERENCES

[1] Rudolf Ahlswede. The capacity region of a channel with two senders and two receivers. *Annals Probability*, 2:805–814, 1974.

[2] Aydano Carleial. Interference channels. *IEEE Transactions on Information Theory*, 24:60–70, 1978.

[3] Aydano Carleial. Outer bounds on the capacity of interference channels. *IEEE Transactions on Information Theory*, 29:602–606, 1983.

[4] Cheng Chang, Raul Etkin, and Erik Ordentlich. Interference channel capacity region for randomized fixed-composition codes. *HP Labs Tech Report*, HPL-2008-194R1, 2008. `http://www.hpl.hp.com/techreports/2008/HPL-2008-194R1.pdf`.

[5] Hon-Fah Chong, Mehul Motani, Hari Krishna Garg, and Hesham El Gamal. On the han-kobayashi region for the interference channel. *IEEE Transactions on Information Theory*, 54:3188–3195, 2008.

[6] Thomas M. Cover and Joy A. Thomas. *Elements of Information Theory, 2nd Edition*. John Wiley and Sons Inc., New York, 2006.

[7] Imre Csiszár. The method of types. *IEEE Transactions on Information Theory*, 44:2505–2523, 1998.

[8] Imre Csiszár and János Körner. Graph decomposition: A new key to coding theorems. *IEEE Transactions on Information Theory*, 27:5– 12, 1981.

[9] Imre Csiszár and János Körner. *Information Theory*. Akadémiai Kiadó, Budapest, 1986.

[10] Gunter Dueck and János Körner. Reliability function of a discrete memoryless channel at rates above capacity. *IEEE Transactions on Information Theory*, 25:82–85, 1979.

[11] Raul Etkin, Neri Merhav, and Erik Ordentlich. Error exponents of optimum decoding for the interference channel. *ISIT*, 2008.

[12] Raul Etkin, Neri Merhav, and Erik Ordentlich. Error exponents of optimum decoding for the interference channel. *IEEE Transactions on Information Theory*, submitted 2008. `http://arxiv.org/abs/0810.1980`.

[13] Raul Etkin and Erik Ordentlich. Discrete memoryless interference channel: New outer bound. *ISIT*, 2007.

[14] Raul Etkin, David Tse, and Hua Wang. Gaussian interference channel capacity within one bit. *IEEE Transactions on Information Theory*, 54:5534–5562, 2008.

[15] Robert Gallager. Fixed composition arguments and lower bounds to error probability. `http://web.mit.edu/gallager/www/notes/notes5.pdf`.

[16] Robert Gallager. *Information Theory and Reliable Communication*. John Wiley, New York, NY, 1971.

[17] Robert Gallager. Capacity and coding for degraded broadcast channels. *Problemy Peredachi Informatsii*, 10(3):3–14, 1974.

[18] Robert Gallager. A perspective on multiaccess channels. *IEEE Transactions on Information Theory*, 31:124 – 142,, 1985.

[19] Gerhard Kramer. Outer bounds on the capacity of gaussian interference channels. *IEEE Transactions on Information Theory*, 50:581586, 2004.

[20] Yu-Sun Liu and Brian L. Hughes. A new universal random coding bound for the multiple access channel. *IEEE Transactions on Information Theory*, 42:376–386, 1996.

[21] Jutta Pokorny and Hans-Martin Wallmeier. Random coding bound and codes produced by permutations for the multiple-access channel. *IEEE Transactions on Information Theory*, 31:741 – 750, 1985.

[22] Igal Sason. On achievable rate region for the gaussian interference channel under strong interference. *IEEE Transactions on Information Theory*, 50:1345–1356, 2004.

[23] Claude Shannon. Two-way communication channels. *Proc. 4th Berkeley Symp. on Mathematical Statistics and Probabilify*, page 61, 1961.

[24] Te sun Han and Kingo Kobayashi. A new achievable rate region for the interference channel. *IEEE Transactions on Information Theory*, 27:49–60, 1981.