

Joint source-channel with side information coding error exponents

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Abstract—In this paper, we study the upper and the lower bounds on the joint source-channel coding error exponent with decoder side-information. The results in the paper are non-trivial extensions of the Csiszár’s classical paper [6]. Unlike the joint source-channel coding result in [6], it is not obvious whether the lower bound and the upper bound are equivalent even if the channel coding error exponent is known. For a class of channels, including the symmetric channels, we apply a game-theoretic result to establish the existence of a saddle point and hence prove that the lower and upper bounds are the same if the channel coding error exponent is known. More interestingly, we show that encoder side-information does not increase the error exponents in this case.

I. INTRODUCTION

In Shannon’s very first paper on information theory [12], it is established that separate coding is optimal for memoryless source channel pairs. Reliable communication is possible if and only if the entropy of the source is lower than the capacity of the channel. However, the story is different when error exponent is considered. It is shown that joint source-channel coding achieves strictly better error exponent than separate¹ coding [6]. The key technical component of [6] is a channel coding scheme to protect different message sets with different channel coding error exponents. In this paper, we are concerned with the joint source-channel coding with side information problem as shown in Figure 1. For a special setup of Figure 1, where the discrete memoryless channel (DMC) is a noiseless channel with capacity² R , i.e. the source coding with side-information problem, the reliable reconstruction of a^n at the decoder is possible if and only if R is larger than the conditional entropy $H(P_{A|B})$ [14]. The error exponents of this problem is also studied in [9], [7] and more importantly in [1].

The duality between source coding with decoder side-information and channel coding is established in the 80’s [1]. This is an important result that all the channel coding error exponent bounds can be easily applied to source coding with side-information error exponent. The result is a consequence

This work was done when he was a postdoctoral researcher with the Hewlett-Packard Laboratories, Palo Alto, CA.

¹In [6], Csiszár hand-wavily shows that the obvious separate coding scheme is suboptimal in terms achieving the best error exponent. The rather obvious result is rigidly proved in [15].

²In this paper, we use bits and \log_2 , and R is always non-negative.

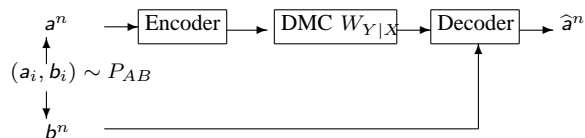


Fig. 1. Source coding with decoder side-information

of the type covering lemma [7], also known as the Johnson-Stein-Lovász theorem [5]. With this duality result, we know that the error exponent of channel coding of channel $W_{Y|X}$ with channel code composition Q_X is essentially the same problem as the error exponent of source coding with decoder side-information where the joint distribution is $Q_X \times W_{Y|X}$. Hence a natural question is what if we put these two dual problems together, what is the error exponent of joint source-channel coding with decoder side-information?

The more general case, where $W_{Y|X}$ is a noisy channel, is recently studied [16], [15]. It is shown that, not surprisingly, the reliable reconstruction of a^n is possible if and only if the channel capacity of the channel is larger than the conditional entropy of the source. A suboptimal error exponent based on a mixture scheme of separate coding and the joint source channel coding first developed in [6] is achieved. In this paper, we follow Csiszár’s idea in [6] and develop a new coding scheme for joint source channel coding with decoder side-information. For a class of channels, including the symmetric channels, the resulted lower and upper bounds have the same property as the joint source-channel coding error exponent *without* side-information in [6]: they match if the channel coding error exponent is known at a critical rate. We use a game theoretic approach to interpret this result.

The outline of the paper is as follows. We review the problem setup and classical error exponent results in Section II. Then in Section III, we present the error exponent result for joint source-channel coding with both decoder and encoder side information which provides a simple upper bound to the error exponent investigated in the paper. This is a simple corollary of Theorem 5 in [6]. The main result of this paper is presented in Section IV. Some implications of these bounds are given in Section V.

II. REVIEW OF SOURCE AND CHANNEL DOING ERROR EXPONENTS

In this paper random variables are denoted by a and b , the realizations of the random variables are denoted by a and b .

A. System model of joint source-channel coding with decoder side-information

As shown in Figure 1, the source and side-information, a^n and b^n respectively, are random variables i.i.d from distribution P_{AB} on a finite alphabet $\mathcal{A} \times \mathcal{B}$. The channel is memoryless with input/output probability transition $W_{Y|X}$, where the input/output alphabets \mathcal{X} and \mathcal{Y} are finite. Without loss of generality, we assume that the number of source symbols and the number of channel uses are equal, i.e. the encoder observes a^n and sends a codeword $x^n(a^n)$ of length n to the channel, the decoder observes the channel output y^n and side-information b^n which is not available to the encoder, the estimate is $\hat{a}^n(b^n, y^n)$.

The error probability is the expectation of the decoding error average over all channel and source behaviors.

$$\Pr(\hat{a}^n \neq \hat{a}^n(b^n, y^n)) = \sum_{a^n, b^n} P_{AB}(a^n, b^n) \sum_{y^n} W_{Y|X}(y^n | x^n(a^n)) 1(a^n \neq \hat{a}^n(b^n, y^n)). \quad (1)$$

The error exponent, for the optimal coding scheme, is defined as $E(P_{AB}, W_{Y|X}) =$

$$\liminf_{n \rightarrow \infty} -\frac{1}{n} \log \Pr(\hat{a}^n \neq \hat{a}^n(b^n, y^n)). \quad (2)$$

The main result of this paper is to establish both upper and lower bounds on $E(P_{AB}, W_{Y|X})$ and show the tightness of these bounds.

B. Classical error exponent results

³ We review some classical results on channel coding error exponents and source coding with side-information error exponents. These bounds are investigated in [10], [7], [9] and [8].

1) *Channel coding error exponents* $E_c(R, W_{Y|X})$: Channel coding is a special case of joint source-channel coding with side-information: the source a and the side-information b are independent, i.e. $P_{AB} = P_A \times P_B$, and a is a uniform distributed random variable on $\{1, 2, \dots, 2^R\}$. For the sake of simplicity, we assume that 2^R is an integer. This is not a problem if 2^R is not an integer since we can lump K symbols together and approximate 2^{KR} by an integer for some K , this is not a problem because $\lim_{K \rightarrow \infty} \frac{1}{K} \log_2(\lfloor 2^{KR} \rfloor) = R$. With this interpretation of channel coding, the definitions of error probability in (1) and error exponent in (2) still holds.

The channel coding error exponent $E_c(R, W_{Y|X})$ is lower bounded by the random coding error exponent and upper

³In this paper, we write the error exponents (both channel coding and source coding) in the style of Csiszár's method of types, equivalent Gallager style error exponents can be derived through the Fenchel duality.

bounded by the sphere packing error exponent.

$$E_r(R, W_{Y|X}) \leq E_c(R, W_{Y|X}) \leq E_{sp}(R, W_{Y|X}) \quad (3)$$

where $E_r(R, W_{Y|X})$

$$\begin{aligned} &= \max_{S_X} \inf_{V_{Y|X}} D(V_{Y|X} \| W_{Y|X} | S_X) + \\ &\quad |I(V_{Y|X}; S_X) - R|^+ \\ &= \max_{S_X} E_r(R, S_X, W_{Y|X}) \end{aligned} \quad (4)$$

and $E_{sp}(R, W_{Y|X})$

$$\begin{aligned} &= \max_{S_X} \inf_{V_{Y|X}: I(V_{Y|X}; S_X) < R} D(V_{Y|X} \| W_{Y|X} | S_X) \\ &= \max_{S_X} E_{sp}(R, S_X, W_{Y|X}) \end{aligned} \quad (5)$$

Here S_X is the input composition (type) of the code words. $E_r(R, W_{Y|X}) = E_{sp}(R, W_{Y|X})$ in the high rate regime that $R > R_{cr}$ where R_{cr} is defined in [10] as the minimum rate for which the sphere packing $E_{sp}(R, W_{Y|X})$ and random coding error exponents $E_r(R, W_{Y|X})$ match for channel $W_{Y|X}$. There are tighter bounds on the channel coding error exponents $E_c(R, W_{Y|X})$ in the low rate regime for $R < R_{cr}$, known as straight-line lower bounds and expurgation upper bounds [10]. However, in this paper, we focus on the basic random coding and sphere packing bounds, as the main message can be effectively carried out.

It is well known [10] that both the random coding and the sphere-packing bounds are decreasing with R and are convex in R . And they are both positive if and only if $R < C(W_{Y|X})$, where $C(W_{Y|X})$ is the capacity of the channel $W_{Y|X}$.

2) *Source coding with decoder side-information error exponents*: This is also a special case of the general setup in Figure 1. This time the channel $W_{Y|X}$ is a noiseless channel with input-output alphabet $\mathcal{X} = \mathcal{Y}$ and $|\mathcal{X}| = 2^R$. Again, we can reasonably assume that 2^R is an integer.

The source coding with side-information error exponent⁴ $e(R, P_{AB})$ can be bounded as follows:

$$e_L(R, P_{AB}) \leq e(R, P_{AB}) \leq e_U(R, P_{AB}), \quad (6)$$

where

$$\begin{aligned} e_L(R, P_{AB}) &= \inf_{Q_{AB}} D(Q_{AB} \| P_{AB}) + |R - H(Q_{A|B})|^+ \\ \text{and } e_U(R, P_{AB}) &= \inf_{Q_{AB}: H(Q_{A|B}) > R} D(Q_{AB} \| P_{AB}). \end{aligned}$$

The duality between channel coding and source coding with decoder side information had been well understood [1]. We give the following duality results on error exponents.

$$e(R, Q_A, P_{B|A}) = E_c(H(Q_A) - R, Q_A, P_{B|A}),$$

or equivalently:

$$e(H(Q_A) - R, Q_A, P_{B|A}) = E_c(R, Q_A, P_{B|A})$$

⁴In this paper, if $R \geq \log_2 |\mathcal{A}|$ for source coding with side-information error exponents, we let the error exponent be ∞ .

where $E_c(R, Q_A, P_{B|A})$ is the channel coding error exponent for channel $P_{B|A}$ at rate R and the codebook composition is Q_A . $e(R, Q_A, P_{B|A})$ is the source coding with side information error exponent at rate R with source sequences uniformly distributed in type Q_A and the side information is the output of channel $P_{B|A}$ with input sequence of type Q_A . So obviously, we have:

$$E_c(R, P_{B|A}) = \max_{Q_A} \{E_c(R, Q_A, P_{B|A})\}$$

$$e(R, P_{AB}) = \min_{Q_A} \{D(Q_A \| P_A) + e(R, Q_A, P_{B|A})\}$$

These results are established by the type covering lemma [6] on the operational level, i.e. a complete characterizations of the source coding with side information error exponent $e(R, Q_A, P_{B|A})$ implies a complete characterizations of the channel coding error exponent $E_c(H(Q_A) - R, Q_A, P_{B|A})$ and vice versa.

From these duality results, it is well known that both the lower and the upper bounds are increasing with R and are convex in R . And they are both positive if and only if $R > H(P_{A|B})$. The special case of the source coding with decoder side information problem is that the side information is independent of the source, i.e. $P_{AB} = P_A \times P_B$. In this case, the error exponent is completely characterized [7],

$$e(R, P_A) = \inf_{Q_A: H(Q_A) > R} D(Q_A \| P_A) \quad (7)$$

3) *Joint source-channel coding error exponents* [6]: In his seminal paper [6], the joint source-channel coding error exponents is studied. This is yet another special case of the general setup in Figure 1. When a and b are independent, i.e. $P_{AB} = P_A \times P_B$, we can drop all the b terms in (1). Hence the error probability is defined as: $\Pr(\hat{a}^n \neq \tilde{a}^n(y^n)) =$

$$\sum_{a^n} P_A(a^n) \sum_{y^n} W_{Y|X}(y^n | x^n(a^n)) 1(a^n \neq \hat{a}^n(y^n)). \quad (8)$$

Write the error exponent of (8) as $E(P_A, W_{Y|X})$. The lower and upper bounds of the error exponents are derived in [6]. It is shown that:

$$\min_R \{e(R, P_A) + E_{sp}(R, W_{Y|X})\} \leq E(P_A, W_{Y|X})$$

$$\leq \min_R \{e(R, P_A) + E_r(R, W_{Y|X})\} \quad (9)$$

The upper bound is derived by using standard method of types argument. The lower bound is a direct consequence of the channel coding Theorem 5 in [6].

The difference between the lower and upper bounds is in the channel coding error exponent. The joint source channel coding error exponent is ‘‘almost’’ completely characterized because the only possible improvement is to determine the channel coding error exponent which is still not completely characterized in the low rate regime where $R < R_{cr}$. However, let R^* be the rate that minimizes $\{e(R, P_A) + E_r(R, W_{Y|X})\}$, if $R^* \geq R_{cr}$ or equivalently $E_r(R^*, W_{Y|X}) = E_{sp}(R^*, W_{Y|X})$, then we have a complete

characterization of the joint source channel coding error exponent:

$$E(P_A, W_{Y|X}) = e(R^*, P_A) + E_r(R^*, W_{Y|X}). \quad (10)$$

The goal of this paper is to derive a similar result for $E(P_{AB}, W_{Y|X})$ defined in (2) as that for the joint source channel coding in (9) and (10).

4) *A recite of Theorem 5 in [6]*: Given a sequence of positive integers $\{m_n\}$ with $\frac{1}{n} \log m_n \rightarrow 0$ and m_n message sets $\mathcal{A}_1, \dots, \mathcal{A}_{m_n}$ each with size $|\mathcal{A}_i| = 2^{nR_i}$. Then there exists a channel code (f_0, ϕ_0) , where the encoder $f_0 : \bigcup_{i=0}^{m_n} \mathcal{A}_i \rightarrow \mathcal{X}^n$ where $f_0(a) = x^n(a) \in S_X^i$ for $a \in \mathcal{A}_i$ and the decoder $\phi_0 : \mathcal{Y}^n \rightarrow \bigcup_{i=0}^{m_n} \mathcal{A}_i$, write $\phi_0(y^n)$ as $\hat{a}(y^n)$ s.t. for any message $a \in \mathcal{A}_i$, the decoding error

$$p_e(a) = \sum_{y^n} W_{Y|X}(y^n | x^n(a)) 1(a \neq \hat{a}(y^n))$$

$$\leq 2^{n(E_r(R_i, S_X^i, W_{Y|X}) - \epsilon_n)}$$

for every channel $W_{Y|X}$, and $\epsilon_n \rightarrow 0$. In particular, if the channel $W_{Y|X}$ is known to the encoder, each S_X^i can be picked to maximize $E_r(R_i, S_X^i, W_{Y|X})$, hence for each $a \in \mathcal{A}_i$:

$$p_e(a) \leq 2^{n(E_r(R_i, W_{Y|X}) - \epsilon_n)}.$$

This channel coding theorem as Csizsár put it, the ‘‘main result of this paper’’ in [6]. We use this theorem directly in the proof of the lower bound in Proposition 1 and further modify it to show the lower bound in Theorem 1.

III. JOINT SOURCE-CHANNEL CODING ERROR EXPONENT WITH BOTH DECODER AND ENCODER SIDE-INFORMATION

As a warmup to the more interesting scenario where the side-information is not known to the encoder, we present the upper/lower bounds when both the encoder and the decoder know the side-information. This setup is shown in Figure 2.

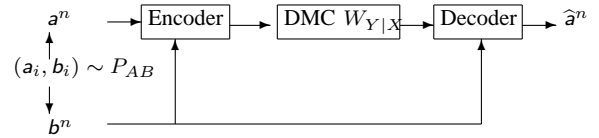


Fig. 2. Source coding with both decoder **and** encoder side-information

The error probability of the coding system is, similar to (1): $\Pr(\hat{a}^n \neq \tilde{a}^n(b^n, y^n)) = \sum_{a^n, b^n} P_{AB}(a^n, b^n)$

$$\sum_{y^n} W_{Y|X}(y^n | x^n(a^n, b^n)) 1(a^n \neq \hat{a}^n(b^n, y^n)). \quad (11)$$

The error exponent of this setup is denoted by $E_{both}(P_{AB}, W_{Y|X})$ which is defined in the same way as $E(P_{AB}, W_{Y|X})$ in (2). The difference is that the encoder observes both source a^n and the side-information b^n , hence the output of the encoder is a function of both: $x^n(a^n, b^n)$.

So obviously, $E_{\text{both}}(P_{AB}, W_{Y|X})$ is not smaller than $E(P_{AB}, W_{Y|X})$.

Comparing (11) and (8), we can see the connections between joint source-channel coding with both decoder and encoder side information and joint source-channel coding. Knowing the side information b^n , the joint source channel coding with both encoder and decoder side information problem is essentially a channel coding problem with messages distributed on \mathcal{A}^n with a distribution $P_{A|B}(a^n|b^n)$. Hence we can extend the results for joint source-channel coding error exponent [6]. We summarize the bounds on $E_{\text{both}}(P_{AB}, W_{Y|X})$ in the following proposition.

Proposition 1: Lower and upper bound on $E_{\text{both}}(P_{AB}, W_{Y|X})$

$$E_{\text{both}}(P_{AB}, W_{Y|X}) \leq \min_R \{e_U(R, P_{AB}) + E_{sp}(R, W_{Y|X})\}$$

$$E_{\text{both}}(P_{AB}, W_{Y|Z}) \geq \min_R \{e_U(R, P_{AB}) + E_r(R, W_{Y|X})\}$$

Not explicitly stated, but it should be clear that the range of R is $(0, \log_2 |\mathcal{A}|)$.

Proof: see Appendix A in [3]. Because $E_{\text{both}}(P_{AB}, W_{Y|X})$ is no smaller than $E(P_{AB}, W_{Y|X})$, so the lower bound of $E(P_{AB}, W_{Y|X})$ in Theorem 1 is also a lower bound for $E_{\text{both}}(P_{AB}, W_{Y|X})$. However, in the appendix in [3], we give a simple proof of the lower bound on $E_{\text{both}}(P_{AB}, W_{Y|X})$ which is a corollary of Theorem 5 in [6]. \square

Comparing the lower and the upper bounds for the case with both encoder and decoder side-information, we can easily see that if R^* minimizes $\{e_U(R, P_{AB}) + E_r(R, W_{Y|X})\}$ and $E_{sp}(R^*, W_{Y|X}) = E_r(R^*, W_{Y|X})$, then the upper bound and the lower bound match. Hence,

$$E_{\text{both}}(P_{AB}, W_{Y|X}) = e_U(R^*, P_{AB}) + E_r(R^*, W_{Y|X}).$$

In this case $E_{\text{both}}(P_{AB}, W_{Y|X})$ is completely characterized.

IV. JOINT SOURCE-CHANNEL ERROR EXPONENTS WITH ONLY DECODER SIDE INFORMATION

We study the more interesting problem where only decoder knows the side-information in this section. We first give a lower and an upper bound on the error exponent of joint source-channel coding with decoder only side-information. The result is summarized in the following Theorem.

Theorem 1: Lower and upper bound on the joint source channel coding with decoder side-information only, as setup in Figure 1, error exponent: For the error probability $\Pr(\hat{a}^n \neq \tilde{a}^n(b^n, y^n))$ and error exponent $E(P_{AB}, W_{Y|X})$ defined in (1)

and (2), we have the following lower and upper bounds:

$$\begin{aligned} E(P_{AB}, W_{Y|X}) &\geq \min_{Q_A} \max_{S_X(Q_A)} \min_{Q_{B|A}, V_{Y|X}} \{ \\ &D(Q_{AB} \| P_{AB}) + D(V_{Y|X} \| W_{Y|X} | S_X(Q_A)) + \\ &|I(S_X(Q_A); V_{Y|X}) - H(Q_{A|B})|^+ \} \\ E(P_{AB}, W_{Y|X}) &\leq \min_{Q_A} \max_{S_X(Q_A)} \\ &\min_{Q_{B|A}, V_{Y|X}: I(S_X(Q_A); V_{Y|X}) < H(Q_{A|B})} \{ D(Q_{AB} \| P_{AB}) \\ &+ D(V_{Y|X} \| W_{Y|X} | S_X(Q_A)) \} \end{aligned}$$

Proof: The main technical tool used here is the method of types. For the lower bound we propose a joint coding scheme for the joint source channel coding with side information problem. This scheme is a modification of the coding scheme first proposed in [6]. However, we cannot directly use the channel coding Theorem 5 in [6] because of the presence of the side information. In essence, we have to study a more complicated case using the method of types. Details see Appendix B in [3]. \square

To simplify the expressions of the lower and upper bounds and later give a sufficient condition for these two bounds to match, we introduce the ‘‘digital interface’’ R and have the following corollary.

Corollary 1: upper and lower bounds on $E(P_{AB}, W_{Y|X})$ with ‘‘digital interface’’ R

$$E(P_{AB}, W_{Y|X}) \leq \min_{Q_A} \max_{S_X(Q_A)} \min_R \{ \quad (12)$$

$$e_U(R, P_{AB}, Q_A) + E_{sp}(R, S_X(Q_A), W_{Y|X}) \}$$

$$E(P_{AB}, W_{Y|Z}) \geq \min_{Q_A} \max_{S_X(Q_A)} \min_R \{ \quad (13)$$

$$e_U(R, P_{AB}, Q_A) + E_r(R, S_X(Q_A), W_{Y|X}) \}$$

where $E_r(R, S_X(Q_A), W_{Y|X})$ is the standard random coding error exponent for channel $W_{Y|X}$ at rate R with input distribution $S_X(Q_A)$ defined in (4), while $e_U(R, P_{AB}, Q_A)$ is a peculiar source coding with side-information error exponent for source P_{AB} at rate R , where the empirical source distribution is fixed at Q_A . That is for Q_A

$$e_U(R, P_{AB}, Q_A) \triangleq \min_{Q_{B|A}: H(Q_{A|B}) \geq R} D(Q_{AB} \| P_{AB})$$

Proof: The proof is in Appendix C in [3]. \square

With the simplified expression of the lower and upper bounds in Corollary 1, we can give a game theoretic interpretation of the bounds. And more importantly, we present some sufficient conditions for the two bounds to match.

A. A game theoretic interpretation of the bounds

The lower and upper bounds established in Corollary 1 clearly have a game theoretic interpretation. This is a two player zero sum game. The first player is ‘‘nature’’, the second player is the coding system, the payoff from ‘‘nature’’ to the coding system is the bounds on the error exponents in Corollary 1. ‘‘Nature’’ chooses the marginal of the source Q_A (observable to the coding system) and R which is essentially

the side information $Q_{B|A}$ and the channel behavior $V_{Y|X}$ (non-observable to the coding system). The coding system choose $S_X(Q_A)$ after observing Q_A . Hence in this game, the "nature" has two moves, the first move on Q_A and the last move on R which is essentially $Q_{B|A}$ and $V_{Y|X}$, while the coding system has the middle move on $S_X(Q_A)$.

Comparing Corollary 1 for joint source-channel coding with decoder side information and the classical joint source-channel coding error exponent [6] in (9), it is desirable to have a sufficient condition that the lower bound and the upper bound match, i.e. the complete characterization as that in (10). It is simpler for the case in (9) since all is needed is that the sphere backing bound and the random coding bound to match at the critical rate R^* as discussed in Section II-B.3. However, for the two bounds in Corollary 1, it is not clear what the conditions are such that these two bounds match. Suppose that the solution of the game (12) is $(Q_A^u, S_X^u(Q_A), R^u)$ and solution of the game (13) is $(Q_A^l, S_X^l(Q_A), R^l)$. An obvious sufficient condition for the two bounds match is as follows:

$$(Q_A^l, S_X^l(Q_A), R^l) = (Q_A^u, S_X^u(Q_A), R^u) \text{ and} \\ E_r(R^u, S_X^u(Q_A), W_{Y|X}) = E_{sp}(R^u, S_X^u(Q_A), W_{Y|X}).$$

This condition is hard to verify for *any* source channel pairs. In the next section, we try to simplify the condition under which these two bounds match for a class of channels.

B. A sufficient condition to reduce $\min\{\max\{\min\{\cdot\}\}\}$ to $\min\{\cdot\}$

The difficulty in studying the bounds in Corollary 1 is that the min and max operators are nested. The problem will be simplified if we can change the order of the min and max operators.

Corollary 2: For symmetric channels $W_{Y|X}$ defined on Page 94 in [10], this includes the binary symmetric and binary erasure channels, where the input distribution S_X to maximize the random coding error exponent $E_r(R, S_X, W_{Y|X})$ is uniform on \mathcal{X} , or for more general channels⁵, where the input distribution S_X to maximize the random coding error exponent $E_r(R, S_X, W_{Y|X})$ is the same for all R , then the upper and lower bounds in Theorem 1 and Corollary 1 can be further simplified to the following forms:

$$E(P_{AB}, W_{Y|X}) \leq \min_R \{e_U(R, P_{AB}) + E_{sp}(R, W_{Y|X})\} \quad (14)$$

$$E(P_{AB}, W_{Y|X}) \geq \min_R \{e_U(R, P_{AB}) + E_r(R, W_{Y|X})\} \quad (15)$$

Note: in this case, the upper and lower bounds for $E(P_{AB}, W_{Y|X})$ is the same as those for $E_{both}(P_{AB}, W_{Y|X})$ in Proposition 1. More discussions see Section V.

Proof: An important property for symmetric channels is that the input distribution that maximizes the random coding

⁵For example, a channel consisted of parallel symmetric channels.

error exponent is constant for all rate R , hence the inner $\max \min\{\cdot\}$ is equal to $\min \max\{\cdot\}$, i.e.

$$\begin{aligned} & E(P_{AB}, W_{Y|X}) \\ & \geq \min_{Q_A} \max_{S_X(Q_A)} \min_R \{e_U(R, P_{AB}, Q_A) + \\ & \quad E_r(R, S_X(Q_A), W_{Y|X})\} \\ & = \min_{Q_A} \min_R \max_{S_X(Q_A)} \{e_U(R, P_{AB}, Q_A) + \\ & \quad E_r(R, S_X(Q_A), W_{Y|X})\} \\ & = \min_{Q_A} \min_R \{e_U(R, P_{AB}, Q_A) + E_r(R, W_{Y|X})\} \quad (16) \\ & = \min_R \{\min_{Q_A} \{e_U(R, P_{AB}, Q_A)\} + E_r(R, W_{Y|X})\} \\ & = \min_R \{e_U(R, P_{AB}) + E_r(R, W_{Y|X})\} \quad (17) \end{aligned}$$

where (16) follows the definition of random coding bound in (3) and (17) follows the obvious equality:

$$\begin{aligned} & \min_{Q_A} e_U(R, P_{AB}, Q_A) = \\ & \min_{Q_{AB}: H(Q_{A|B}) \geq R} D(Q_{AB} \| P_{AB}) = e_U(R, P_{AB}). \end{aligned}$$

The upper bound in 14 is trivial by noticing that $\max \min\{\cdot\} \leq \min \max\{\cdot\}$ [2], hence:

$$\begin{aligned} & E(P_{AB}, W_{Y|X}) \\ & \leq \min_{Q_A} \max_{S_X(Q_A)} \min_R \{e_U(R, P_{AB}, Q_A) + \\ & \quad E_{sp}(R, S_X(Q_A), W_{Y|X})\} \\ & \leq \min_{Q_A} \min_R \max_{S_X(Q_A)} \{e_U(R, P_{AB}, Q_A) + \\ & \quad E_{sp}(R, S_X(Q_A), W_{Y|X})\} \\ & = \min_{Q_A} \min_R \{e_U(R, P_{AB}, Q_A) + E_{sp}(R, W_{Y|X})\} \\ & = \min_R \{\min_{Q_A} \{e_U(R, P_{AB}, Q_A)\} + E_{sp}(R, W_{Y|X})\} \\ & = \min_R \{e_U(R, P_{AB}) + E_{sp}(R, W_{Y|X})\} \quad (18) \end{aligned}$$

Corollary 2 is proved. \square

With this corollary proved, we can give a sufficient condition under which the lower bound and upper bound match similar to that for the joint source-channel coding case in Section II-B.3. More discussions see Section V.

C. Why it is hard to generalize Corollary 2 to non-symmetric channels?

Whether $\max_{S_X(Q_A)} \min_R \{e_U(R, P_{AB}, Q_A) + E_r(R, S_X(Q_A), W_{Y|X})\}$ is equal to $\min_R \max_{S_X(Q_A)} \{e_U(R, P_{AB}, Q_A) + E_r(R, S_X(Q_A), W_{Y|X})\}$ is not obvious for general (non-symmetric) channels. A sufficient condition of the existence of a unique saddle point hence the equality is known as the Sion's Theorem [13] which states that:

$$\max_{\mu \in \mathcal{M}} \min_{\nu \in \mathcal{N}} f(\mu, \nu) = \min_{\nu \in \mathcal{N}} \max_{\mu \in \mathcal{M}} f(\mu, \nu) \quad (19)$$

if \mathcal{M} and \mathcal{N} are convex, compact spaces and f a quasi-concave-convex (definitions see [2]) and continuous function

on $\mathcal{M} \times \mathcal{N}$. For the function of interest:

$$\max_{S_X(Q_A)} \min_R \{e_U(R, P_{AB}, Q_A) + E_r(R, S_X(Q_A), W_{Y|X})\}. \quad (20)$$

We examine the sufficient condition under which a unique equilibrium exists, according to the Sion's Theorem. First, $e_U(R, P_{AB}, Q_A) + E_{sp}(R, S_X(Q_A), W_{Y|X})$ is quasi-convex in R because both $e_U(R, P_{AB}, Q_A)$ and $E_{sp}(R, S_X(Q_A), W_{Y|X})$ are convex, hence quasi-convex in R . However, (20) is not quasi-concave on $S_X(Q_A)$:

$$\begin{aligned} & E_r(R, S_X(Q_A), W_{Y|X}) \\ &= \inf_{V_{Y|X}} D(V_{Y|X} \| W_{Y|X} | S_X(Q_A)) \\ &+ |I(V_{Y|X}; S_X(Q_A)) - R|^+, \end{aligned}$$

notice that the first term is linear in $S_X(Q_A)$, the second term is quasi-concave but not concave. But the sum of a linear function and a quasi-concave function might not be quasi-concave. This shows that the minmax theorem cannot be established by using the Sion's Theorem. This does not mean that the minmax theorem cannot be proved. However for a non quasi-concave function that may have multiple peaks, $\min \max\{\cdot\}$ is not necessarily equal to $\max \min\{\cdot\}$.

V. "ALMOST" COMPLETE CHARACTERIZATION OF $E(P_{AB}, W_{Y|X})$ FOR SYMMETRIC CHANNELS

The sufficient condition in Corollary 2 is important, since binary symmetric and binary erasure channels are among the most well studied discrete memoryless channels. We further discuss the implications of the "almost" complete characterization of $E(P_{AB}, W_{Y|X})$ for symmetric channels.

First we give an example shown in Figure 3 and Figure 4. The source \mathbf{a} is a Bernoulli 0.5 random variable and the joint distribution has the distribution

$$P_{AB} = \begin{Bmatrix} 0.50 & 0.00 \\ 0.05 & 0.45 \end{Bmatrix} \quad (21)$$

The channel $W_{Y|X}$ is a binary symmetric channel with cross rate 0.025. The channel coding error exponent bounds $E_r(R, W_{Y|X})$ and $E_{sp}(R, W_{Y|X})$ and the source coding with decoder side-information upper bound $e_U(R, P_{AB})$ are plotted in Figure 3. The channel coding bound match while $R \geq R_{cr}$, where R_{cr} is defined in [10].

Note: the lower bound of the source coding with side information error exponent $e_L(R, P_{AB})$ is not plotted in the figure.

In Figure 4, we add both the lower and upper bounds on the joint source channel coding with decoder side information to the plot in Figure 3. For this source channel pair P_{AB} and $W_{Y|X}$, we have a complete characterization of $E_{both}(P_{AB}, W_{Y|X})$ because the channel is symmetric and the two bounds match at the minimal point, i.e. the two curves: $e_U(R, P_{AB}) + E_{sp}(R, W_{Y|X})$ and $e_U(R, P_{AB}) + E_r(R, W_{Y|X})$ match at the minimal point as shown in Figure 4. The value of the minimum is E_j shown in Figure 4.

A. Encoder side information often does not help

Similar to Proposition 1, we can see the conditions under which we can give a complete characterization of the joint source channel coding with decoder only side information error exponent $E(P_{AB}, W_{Y|X})$. If R^* minimizes $\{e_U(R, P_{AB}) + E_r(R, W_{Y|X})\}$ and $E_{sp}(R^*, W_{Y|X}) = E_r(R^*, W_{Y|X})$, then the upper bound and the lower bound match. Hence:

$$E(P_{AB}, W_{Y|X}) = e_U(R^*, P_{AB}) + E_r(R^*, W_{Y|X}). \quad (22)$$

Comparing Corollary 2 and Proposition 1, we bound the error exponent with or without decoding side-information by the same lower and upper bounds. This does not mean that $E(P_{AB}, W_{Y|Z}) = E_{both}(P_{AB}, W_{Y|Z})$ always holds. But if the lower bound and upper bound match, which is shown in Figure 4, then we have:

$$\begin{aligned} E(P_{AB}, W_{Y|Z}) &= E_{both}(P_{AB}, W_{Y|Z}) = \\ &e_U(R^*, P_{AB}) + E_r(R^*, W_{Y|X}). \end{aligned}$$

where R^* minimizes $e_U(R, P_{AB}) + E_r(R, W_{Y|X})$ and $R^* > R_{cr}$. This is another example for block coding where knowing side-information does not help increase the error exponent. In the contrary, as discussed in [4], in the delay constrained setup, there is a penalty for not knowing the side-information even if the channel is noiseless.

B. Separate coding is strictly sub-optimal

An obvious coding scheme for the problem in Figure 1 is to implement a separate coding scheme. A source encoder first encodes the source sequence \mathbf{a}^n into a rate R , where R is determined later, bit stream $c^{nR}(\mathbf{a}^n)$ then an independent channel encoder encodes the bits c^{nR} into channel inputs \mathbf{x}^n . The channel decoder first decodes the channel output \mathbf{y}^n into bits \hat{c}^{nR} and then the independent source decoder reconstructs $\hat{\mathbf{a}}^n$ from \hat{c}^{nR} and side information \mathbf{b}^n . This is a separate coding scheme with outer source with side information coding and inner channel coding, both at rate R . If both coding are random coding that achieves the random coding error exponents for both source coding and channel coding respectively. The union bound of the error probability is as follows:

$$\begin{aligned} & \Pr(\mathbf{a}^n \neq \hat{\mathbf{a}}^n(\mathbf{b}^n, \mathbf{y}^n)) \\ &= \Pr(c^{nR} \neq \hat{c}^{nR}(\mathbf{y}^n)) + \Pr(\mathbf{a}^n \neq \hat{\mathbf{a}}(\hat{c}^{nR}(\mathbf{y}^n), \mathbf{b}^n), \\ & \quad c^{nR} = \hat{c}^{nR}(\mathbf{y}^n)) \quad (23) \\ &\leq \Pr(c^{nR} \neq \hat{c}^{nR}(\mathbf{y}^n)) + \Pr(\mathbf{a}^n \neq \hat{\mathbf{a}}(\hat{c}^{nR}(\mathbf{y}^n), \mathbf{b}^n) \\ & \quad | c^{nR} = \hat{c}^{nR}(\mathbf{y}^n)) \quad (24) \\ &\leq 2^{-n(E_r(R, W_{Y|X}) - \epsilon_n^1)} + 2^{-n(e_L(R, P_{AB}) - \epsilon_n^2)} \quad (25) \end{aligned}$$

where ϵ_n^1 and ϵ_n^2 converges to zero as n goes to infinity. (23) follows the union bound argument that a decoding error occurs if either the inner channel coding fails or the outer source coding fails. (24) is true because conditional probability is large or equal to joint probability. Finally (25) is true because both the outer source coding and inner channel coding achieve

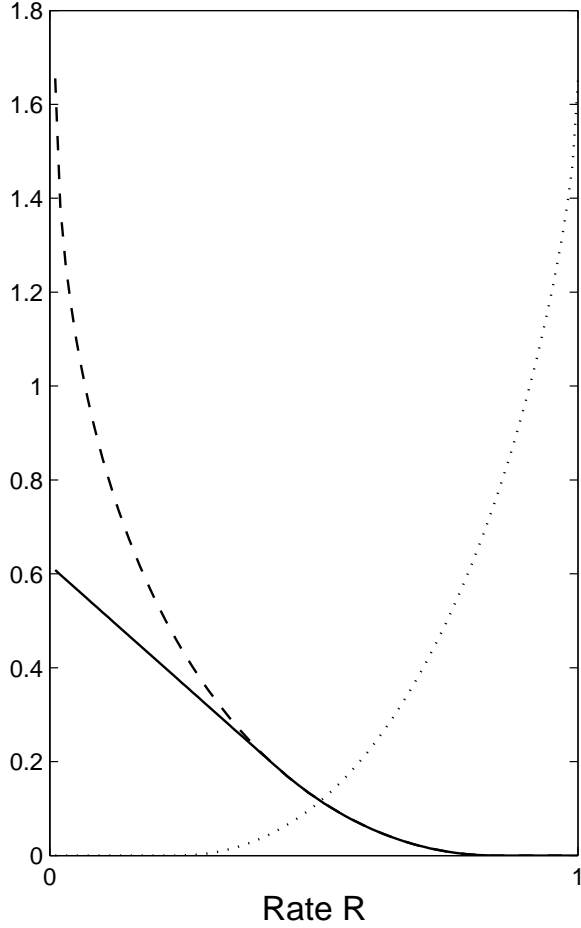


Fig. 3. The upper bound on source coding with side-information error exponent $e_U(R, P_{AB})$ is the dotted line. The random coding bound $E_r(R, W_{Y|X})$ and sphere packing bound $E_{sp}(R, W_{Y|X})$ for channel coding error exponents are the solid line and the dashed line respectively.

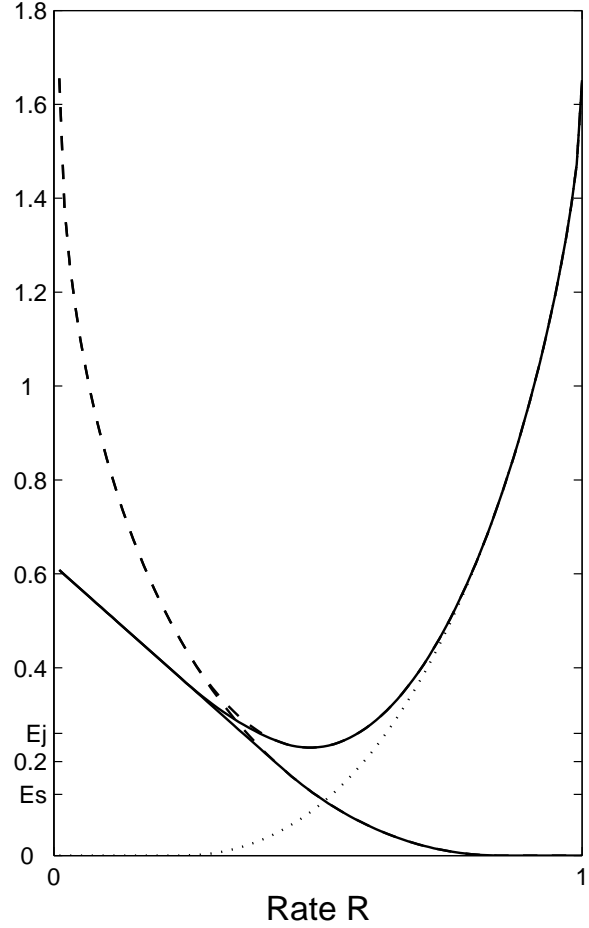


Fig. 4. $e_U(R, P_{AB}) + E_{sp}(R, W_{Y|X})$ and $e_U(R, P_{AB}) + E_r(R, W_{Y|X})$ are added to Figure 3 in dashed line and solid line respectively, they match at the minimal point hence the joint source-channel coding with decoder side-information error exponent is completely determined as $E(P_{AB}, W_{Y|X}) = E_j$ And E_s is the separate coding error exponent $E_{separate}(P_{AB}, W_{Y|X})$ defined in (26).

the random coding error exponents. From (25) and that we can optimize the digital interface rate R between the channel coder and source coder, we know that the separate coding error exponent is

$$\begin{aligned} & \max_R \{ \min \{ E_r(R, W_{Y|X}), e_L(R, P_{AB}) \} \} \\ & \triangleq E_{separate}(P_{AB}, W_{Y|X}) \end{aligned} \quad (26)$$

This separate coding scheme is also discussed for joint source channel coding in [6]. A similar bound is drawn. We next show why the separate coding error exponent $E_{separate}(P_{AB}, W_{Y|X})$ is in general strictly smaller than the lower bound of $E(P_{AB}, W_{Y|X})$ in (15).

First, obviously, $E_{separate}(P_{AB}, W_{Y|X}) \leq \max_R \{ \min \{ E_r(R, W_{Y|X}), e_U(R, P_{AB}) \} \}$. Secondly

$\{E_r(R, W_{Y|X})$ is monotonically decreasing, $e_U(R, P_{AB})$ is monotonically increasing, and both are continuous and convex as shown in Figure 4. This means that for rate \bar{R} such that $E_r(\bar{R}, W_{Y|X}) = e_U(\bar{R}, P_{AB})$:

$$E_{separate}(P_{AB}, W_{Y|X}) = E_r(\bar{R}, W_{Y|X}) = e_U(\bar{R}, P_{AB})$$

Now let R^* be the rate to minimize $\{e_U(R, P_{AB}) + E_r(R, W_{Y|X})\}$, i.e.

$$E(P_{AB}, W_{Y|X}) \geq e_U(R^*, P_{AB}) + E_r(R^*, W_{Y|X}).$$

There are three scenarios. First if $R^* = \bar{R}$, then

$$\begin{aligned} E(P_{AB}, W_{Y|X}) & \geq e_U(R^*, P_{AB}) + E_r(R^*, W_{Y|X})X \\ & = 2E_r(\bar{R}, W_{Y|X}) = 2E_{separate}(P_{AB}, W_{Y|X}). \end{aligned}$$

Secondly, if $R^* < \bar{R}$,

$$\begin{aligned} E(P_{AB}, W_{Y|X}) &\geq E_r(R^*, W_{Y|X}) > E_r(\bar{R}, W_{Y|X}) \\ &= E_{\text{separate}}(P_{AB}, W_{Y|X}). \end{aligned}$$

Finally if $R^* > \bar{R}$,

$$\begin{aligned} E(P_{AB}, W_{Y|X}) &\geq e_U(R^*, P_{AB}) > e_U(\bar{R}, P_{AB}) \\ &= E_{\text{separate}}(P_{AB}, W_{Y|X}). \end{aligned}$$

So in all cases, the joint source channel coding error exponent $E(P_{AB}, W_{Y|X})$ is strictly larger than the separate coding error exponent $E_{\text{separate}}(P_{AB}, W_{Y|X})$. This is clearly illustrated in Figure 4.

Note: $E_{\text{separate}}(P_{AB}, W_{Y|X})$ is an achievable separate coding error exponent from the obvious separate coding scheme. What we prove is that this obvious one is strictly smaller than the joint source-channel coding error exponent. This is similar to the claim Csiszár makes in [6]. It should be clear that the upper bound of any separate source channel coding error exponent is $\max_R \{\min\{E_{\text{sp}}(R, W_{Y|X}), e_U(R, P_{AB})\}\}$ which is comparable to (26). The proof hinges on the complete transparency between the source coding and channel coding, otherwise we have a joint coding schemes. A detailed discussion is in [15].

ACKNOWLEDGEMENT

The author would like to thank Neri Merhav and Filippo Balestrieri for pointing out the Sion's theorem in [13], and Fady Alajaji for many helpful discussions on the technical results of this paper.

VI. CONCLUSIONS

We study the joint source channel coding with decoder side-information problem, with or without encoder side-information. This is an extension of Csiszár's joint source channel coding error exponent problem in [6]. To derive the lower bound, we use a novel joint source channel with decoder side-information decoding scheme. We further investigate the conditions under which the lower bounds and upper bounds match. A game theoretic approach is applied to show the equivalence of the lower and upper bound. This approach might be useful in simplifying other error exponents with a cascade of min-max operators, for example, the Wyner-Ziv coding error exponent recently studied in [11].

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